Math 509 **Dr. DeTurck**
Assignment 4 **Due Thursday, February 12**

1. Let $A \subset \mathbf{R}^2$ be the annulus (ring-shaped region)
$$A = \{(x,y) \in \mathbf{R}^2 \,:\, 1 < x^2 + y^2 < 9\}$$
Suppose $f : A \to \mathbf{R}^1$ is a differentiable function that has the property that $\|Df\| < 3$ at every point of the annulus A. Furthermore, suppose $f(2,0) = 9$. Give the best estimate you can for $f(-2,0)$.

2. Find and classify all the critical points of the function
$$f(x,y) = (x^2+y^2)e^{\frac{-(x^2+4y^2)}{9}}.$$

3. Show that the function $g(x,y) = (x-y^2)(x-3y^2)$ has the property that its restriction to any line through the origin has an absolute local minimum at $(0,0)$, yet g does not have a local minimum at $(0,0)$ (or anywhere else, for that matter).

4. Where (i.e., in neighborhoods of what points) do the equations
$$x+y+z=0$$
$$x^2+y^2+z^2+2xz-1=0$$
determine x and y as functions of z?

5. Show that the function f defined by $f(t) = t + 2t^2\cos(1/t)$ for $t \neq 0$ and $f(0) = 0$ is differentiable on $\mathbf{R}$ and $f'(0) = 1 \neq 0$, but there is no neighborhood of $t = 0$ on which f is one-to-one. Why does this not contradict the inverse function theorem?

6. Are the functions
$$u_1 = \frac{x+y}{x-y} \qquad u_2 = \frac{xy}{(x-y)^2}$$
functionally dependent (i.e., can one be written as a function of the other)? How does the derivative help you decide this? How about the functions
$$u_1 = \frac{yz-x}{x} \qquad u_2 = \frac{xyz - y^2z^2 + x^2}{xyz + y^2z^2}\ ?$$

7. (*Lagrange multipliers I*) Let Ω be an open subset of $\mathbf{R}^m$ and suppose that $\varphi\colon \Omega \to \mathbf{R}$ and $f\colon \Omega \to \mathbf{R}$ are continuously differentiable on Ω and that the point $c = (c_1, \ldots, c_n)$ is in the set $M = \{x \in \Omega : f(x) = 0\}$. Further, suppose that $Df \neq 0$ throughout Ω. Finally, assume that the restriction of φ to M has a local extremum (max or min) at c – this means that there is some neighborhood V of c so that either $\varphi(x) \leq \varphi(c)$ for all $x \in V \cap M$ or $\varphi(x) \geq \varphi(c)$ for all $x \in V \cap M$. Show that there is a (unique) number λ such that

$$\frac{\partial \varphi}{\partial x_i}(c) + \lambda \frac{\partial f}{\partial x_i}(c) = 0. \qquad (*)$$

The point of this is that if one seeks points c at which $\varphi|_M$ has local extrema, then one need only search among those $c \in M$ for which this last equation has a solution λ (so one need only consider solutions $c_1, \ldots, c_n, \lambda$ of $\partial f / \partial x_i = 0,\ i = 1, \ldots, n$ and (*)). The variable λ is known as a *Lagrange multiplier*.