Math 509 **Dr. DeTurck**
Assignment 5 **Due Thursday, February 19**

1. (*Lagrange multipliers II*) Let Ω be an open subset of $\mathbf{R}^m$ and suppose that $\varphi\colon\Omega\to\mathbf{R}$ and $f\colon\Omega\to\mathbf{R}^n$ (with $n<m$) are continuously differentiable on Ω and that the point $c=(c_1,\ldots,c_m)$ is in the set $M=\{x\in\Omega\,:\,f(x)=0\in\mathbf{R}^n\}$. Further, suppose that Df has rank n throughout Ω (or, to simplify, assume that the $n\times n$ matrix whose ij-element is $\partial f_i/\partial x_j$ is nonsingular). Finally, assume that the restriction of φ to M has a local extremum (max or min) at c – this means that there is some neighborhood V of c so that either $\varphi(x)\le\varphi(c)$ for all $x\in V\cap M$ or $\varphi(x)\ge\varphi(c)$ for all $x\in V\cap M$. Show that there is a (unique) vector $\lambda=(\lambda_1,\ldots,\lambda_n)$ such that

$$\frac{\partial\varphi}{\partial x_j}(c)+\sum_{i=1}^{n}\lambda_i\frac{\partial f_i}{\partial x_j}(c)=0.\qquad(1\le j\le m)\qquad(*)$$

The point of this is that if one seeks points c at which $\varphi|_M$ has local extrema, then one need only search among those $c\in M$ for which the system (*) has a solution λ (so one need only consider solutions $c_1,\ldots,c_m,\lambda_1,\ldots,\lambda_n$ of the system of $m+n$ equations $\partial f/\partial x_i=0,\ i=1,\ldots,n$ and (*)). The variables $\lambda_1,\ldots,\lambda_n$ are known as *Lagrange multipliers*.

2. Use Lagrange multipliers I and/or II to solve the following:

(a) Find the point on the hyperplane $M=\{x\in\mathbf{R}^n\mid\sum_{i=1}^n a_ix_i=b\}$ that is closest to the origin in $\mathbf{R}^n$. (Consider the function $\varphi(x)=|x|^2$ restricted to the hyperplane.)

(b) Prove that if $x_1,\ldots,x_n$ are numbers for which $\sum_{i=1}^n x_i^2=1$ (in other words, $(x_1,\ldots,x_n)$ is a point on the unit sphere in $\mathbf{R}^n$, then

$$\prod_{i=1}^{n}x_i^2\le\frac{1}{n^n},$$

and equality occurs if and only if $x_i^2=1/n$ for all i.

(c) Prove that if $r_1,\ldots,r_n$ are non-negative real numbers, then

$$\left(\prod_{i=1}^{n}r_n\right)^{1/n}\le\frac{1}{n}\sum_{i=1}^{n}r_i,$$

and equality occurs if and only if $a_i=a_1$ for all i. This is called the *arithmetic-geometric mean inequality.* (Set $x_i^2=a_i/\sum a_j$ in (b)).

3. The object of this exercise is to show that the orthogonal group $O(n)$, consisting of $n \times n$ (real) matrices M satisfying $MM^t = I$ has precisely two connected components. Recall that in class, we showed that the $O(n)$ is a compact subset of $\mathbf{R}^{n^2}$ and that the determinant of an orthogonal matrix is ± 1 (so there are *at least* two connected components). In this problem, we'll write $\langle \mathbf{x}, \mathbf{y} \rangle$ for the dot product of the vectors $\mathbf{x}$ and $\mathbf{y}$.

(a) Even though we did this in class, show that if $M \in O(n)$, then $\langle M\mathbf{x}, M\mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle$ for all vectors $\mathbf{x}$ and $\mathbf{y}$.

(b) Let $\mathbf{x}$ and $\mathbf{y}$ be any non-antipodal pair of points on the unit sphere $S^{n-1} \subset \mathbf{R}^n$, so $\langle \mathbf{x}, \mathbf{x} \rangle = \langle \mathbf{y}, \mathbf{y} \rangle = 1$ and $\mathbf{y} \notin \{\mathbf{x}, -\mathbf{x}\}$. Show that the vector $\mathbf{v} = \mathbf{y} - \langle \mathbf{x}, \mathbf{y} \rangle \mathbf{x}$ is nonzero, and is perpendicular to $\mathbf{x}$, and $\langle \mathbf{v}, \mathbf{v} \rangle^2 = 1 - \langle \mathbf{x}, \mathbf{y} \rangle^2$.

(c) Let α be defined via $\cos \alpha = \langle \mathbf{x}, \mathbf{y} \rangle$. Show that the curve

$$G(\sigma) = \cos \sigma \, \mathbf{x} + \sin \sigma \left(\frac{\mathbf{y} - \langle \mathbf{x}, \mathbf{y} \rangle \mathbf{x}}{|\mathbf{y} - \langle \mathbf{x}, \mathbf{y} \rangle \mathbf{x}|} \right), \qquad 0 \le \sigma \le \alpha$$

satisfies $G(0) = \mathbf{x}$, $G(\alpha) = \mathbf{y}$ and $\langle G(\sigma), G(\sigma) \rangle = 1$ for all $\sigma \in [0, \alpha]$. Also, show that the arc-length of the curve $G(\sigma$ is α. (In the language of differential geometry, $G(\sigma)$ is the unique shortest geodesic connecting $\mathbf{x}$ to $\mathbf{y}$ on S^{n-1}, and has length α). We can rewrite this as

$$G(\sigma) = \cos \sigma \, \mathbf{x} + \sin \sigma \left(\frac{\mathbf{y} - \cos \alpha \, \mathbf{x}}{\sin \alpha} \right).$$

(d) If $\mathbf{x}$ and $\mathbf{y}$ are non-antipodal points in $S^{n-1} \subset \mathbf{R}^n$, in other words if $\mathbf{x}$ and $\mathbf{y}$ are linearly independent unit vectors in $\mathbf{R}^n$, then there is a unique element $M \in SO(n)$ (i.e., an orthogonal map with determinant $+1$) that maps $\mathbf{y}$ to $\mathbf{x}$ and leaves fixed the $n-2$-dimensional subspace of $\mathbf{R}^n$ consisting of all vectors perpendicular to both $\mathbf{x}$ and $\mathbf{y}$. Show that for $\mathbf{v} \in \mathbf{R}^n$, this mapping is given by

$$M\mathbf{v} = \mathbf{v} - \frac{\langle \mathbf{x} + \mathbf{y}, \mathbf{v} \rangle}{1 + \langle \mathbf{x}, \mathbf{y} \rangle} \mathbf{y} + \left\langle \frac{1 + 2\langle \mathbf{x}, \mathbf{y} \rangle}{1 + \langle \mathbf{x}, \mathbf{y} \rangle} \, \mathbf{y} - \frac{1}{1 + \langle \mathbf{x}, \mathbf{y} \rangle} \, \mathbf{x} \, , \, \mathbf{v} \right\rangle \mathbf{x}.$$

(e) If $\mathbf{x}$ and $\mathbf{y}$ happen to be antipodal points of S^{n-1} for $n \ge 3$, then there are many elements of $SO(n)$ that map $\mathbf{x}$ to $\mathbf{y}$. Prove this (i.e., find at least two different ones).

(f) If $\mathbf{x}$ and $\mathbf{y}$ are any points on $S^{n-1} \subset \mathbf{R}^n$, show that there is a continuous one-parameter family of elements of $SO(n)$, i.e., a continuous map from the interval $[0,1]$ to $SO(n)$, say R_t, such that R_0 is the identity matrix (so $R_0(\mathbf{x}) = \mathbf{x}$) and $R_1(\mathbf{x}) = \mathbf{y}$. (You're all set up to do this using parts (c) and (d) – if $\mathbf{x}$ and $\mathbf{y}$ are antipodal you have to make an intermediate step.)

(g) Let $M \in SO(n)$. Show that there is a one-parameter family of elements of $SO(n)$, say M_t such that $M_0 = M$ and $M_1(\mathbf{e}_1) = \mathbf{e}_1$ (where $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ is the standard orthonormal basis of $\mathbf{R}^n$). (Write $M_t = R_t M_0$ and use part (f).)

(h) Explain why the matrix of M_1 in part (g) has the form:

$$\left[\begin{array}{c|c} 1 & 0^{1\times n} \\ \hline 0^{n\times 1} & B_1 \end{array}\right]$$

where $B_1 \in SO(n-1)$ (what are $\langle \mathbf{e}_1, M_1(\mathbf{e}_2)\rangle, \ldots \langle \mathbf{e}_1, M_1(\mathbf{e}_n)\rangle$?).

(i) Explain how to successively connect M to matrices of the form

$$\left[\begin{array}{c|c} I^{p\times p} & 0^{p\times n} \\ \hline 0^{n\times p} & B_p \end{array}\right]$$

where $B_p \in SO(n-p)$, and so ultimately to the $n \times n$ identity matrix.

(j) Explain why the conclusion of part (i) is that $SO(n)$ is connected. Why does this prove that $O(n)$ has precisely two connected components?

4. Explain how to modify the reasoning in the preceding problem to show that $GL(n)$ has precisely two connected components (the connected component that contains the identity matrix is called $SL(n)$, for "special linear" group).