Math 509 **Dr. DeTurck**
Assignment 6 **Due Thursday, March 26**

Let's start with a couple of problems from the current *College Math Journal*.

1. Let f be a function defined on an open interval I containing the interval $[a, b]$, and such that $f''(x) \geq 0$ for all $x \in I$. Show that

$$\int_0^1 f(a + (b-a)y)\, dy \geq \int_0^1 f\left(\frac{3a+b}{4} + \frac{b-a}{2}y\right) dy.$$

2. Let k be a positive integer. Show that

$$\lim_{n\to\infty} n^2 \int_0^1 \frac{x^n}{1 + x^k + x^{2k} + \cdots + x^{nk}}\, dx = k \sum_{m=0}^{\infty} \frac{1}{(1+km)^2}.$$

3. Let a, b, and c be positive real numbers with $a + b + c = 3$. Show that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq 1 + 2\sqrt{\frac{a^2+b^2+c^2}{3abc}}.$$

4. Evalute the integral

$$\int_0^\infty\int_0^\infty \frac{x^2+y^2}{1+(x^2-y^2)^2} e^{-2xy}\, dx\, dy$$

using the change of variables $u = x^2 - y^2$, $v = 2xy$.

5. Evaluate

$$\iint_T \left(\frac{x-y}{x+y}\right)^2 dx\, dy$$

where T is the triangle with vertices $(0,0)$, $(0,1)$, $(1/2, 1/2)$ by using an appropriate change of variables.

6. A k-form ω is called *decomposable* if there are one-forms $\alpha_1, \ldots, \alpha_k$ such that $\omega = \alpha_1 \wedge \cdots \wedge \alpha_k$.

(a) If $\varphi_1, \ldots, \varphi_4$ is a basis for $(\mathbf{R}^4)^*$, show that $\varphi_1 \wedge \varphi_2 + \varphi_3 \wedge \varphi_4$ is *not* decomposable.

(b) Show that every 2-form on $\mathbf{R}^3$ is decomposable.

(c) Show that every $n-1$-form on $\mathbf{R}^n$ is decomposable.

7. Let $\alpha \in \Lambda^p V^*$, where V is a vector space of dimension n and $\alpha \neq 0$.

   (a) Show that the set of one-forms $\omega \in V^*$ such that $\alpha \wedge \omega = 0$ is a subspace of V^*.

   (b) Show that the dimension of this subspace is at most p. Can it be strictly less than p? What can you conclude about α if the dimension is exactly p? (Try the cases $p = 1, 2$ first).

8. The *kernel* of a 2-form $\omega \in \Lambda^2 V^*$ is defined to be

$$\ker(\omega) = \{\mathbf{v} \in V \,|\, \omega(\mathbf{v}, \mathbf{w}) = 0 \text{ for all } \mathbf{w} \in V\}.$$

   (a) Show that on $\mathbf{R}^4$, where $\mathbf{v}_1, \ldots, \mathbf{v}_4$ is a basis for $\mathbf{R}^4$ and $\varphi_1, \ldots, \varphi_4$ is the dual basis for $(\mathbf{R}^4)^*$, that $\ker(\varphi_1 \wedge \varphi_2)$ is the span of $\{\mathbf{v}_3, \mathbf{v}_4\}$ and $\ker(\varphi_1 \wedge \varphi_2 + \varphi_3 \wedge \varphi_4) = \{0\}$.

   (b) More generally, show that the kernel of a 2-form on any V has even codimension. (Infer that the kernel of any 2-form on an odd-dimensional space is non-trivial.)

9. Let $\alpha_1, \ldots, \alpha_n$ be independent 1-forms. Let $\{\varphi_{ij} \,|\, i, j = 1, \ldots, n\}$ be a set of 1-forms with $\varphi_{ij} = -\varphi_{ji}$. Show that if

$$\sum_{i=1}^{n} \alpha_i \wedge \varphi_{ij} = 0$$

   for all $j = 1, \ldots, n$, then each $\varphi_{ij} = 0$.

10. Let $\alpha_1, \ldots, \alpha_k$ be independent 1-forms. Suppose $\beta_1, \ldots, \beta_k$ are 1-forms such that

$$\alpha_1 \wedge \beta_1 + \cdots + \alpha_k \wedge \beta_k = 0.$$

   Show that there are constants c_{ij}, $i, j = 1, \ldots, k$ such that

$$\beta_i = \sum_{j=1}^{k} c_{ij} \alpha_j$$

   and $c_{ij} = c_{ji}$. (This is called *Cartan's lemma*).