Math 509 **Dr. DeTurck**
Assignment 7 **Due Thursday, April 2**

OK, we've been developing all this differential form machinery, and we're at a somewhat awkward place in the sense that there's not much we can do with it yet.

On Tuesday, we'll define what are called "singular cubes" and "singular chains" and how to integrate differential forms on them (it turns out that just about any integral you'd want to do is an integral over a singular chain).

Even though the terminology might sound weird, here's how it goes: The *standard k-cube in $\mathbf{R}^k$* is simply the subet

$$I_k = [0,1]^k = \{(x_1, \ldots, x_k) \in \mathbf{R}^k \,|\, 0 \le x_i \le 1,\ i = 1, \ldots, k\}.$$

That's simple enough. Next, a *singular k-cube in $\mathbf{R}^n$* is simply a (smooth) mapping c from the standard k-cube in $\mathbf{R}^k$ into $\mathbf{R}^n$. For instance, a singluar 1-cube is just a curve, etc.

We'll define integration of forms as follows: if $\omega = f\,dx^1 \wedge \cdots \wedge dx^k$ is a k-form on I_k, then

$$\int_{I_k} \omega = \int_{I_k} f = \int_{[0,1]^k} f(x_1, \ldots, x_k)\,dx_1 \cdots dx_k.$$

And then it's natural to define the integral of a k-form ω on $\mathbf{R}^n$ over a singular k-cube $c : I_k \to \mathbf{R}^n$ as follows:

$$\int_c \omega = \int_{I_k} c^*\omega.$$

Basically, this just means the usual parametrize-substitute-integrate drill of multivariate calculus, except we have to be careful about signs in the differential forms – for instance

$$\int_{I_2} dx_2 \wedge dx_1 = -1.$$

Having these definitions (and just one more!) gives us precisely the set-up we need to prove Stokes's Theorem in all its glorious generality.

OK, now for some problems:

1. (a) Let $c_1 : I_1 \to \mathbf{R}^2$ be given by $c_1(t) = (t, t^2)$ for $0 \le t \le 1$, and let $\omega = x_2 \sin x_1\,dx_1 + 3(x_2)^2\,dx^2$. Calculate

$$\int_{c_1} \omega.$$

(b) Let $c_2 \colon I_1 \to \mathbf{R}^2$ be given by $c_2(t) = (t^2, t^4)$. for $0 \le t \le 1$. Calculate

$$\int_{c_2} \omega$$

for ω as in part (a).

(c) What is the relationship between parts (a) and (b)? Formulate and prove a theorem (something about reparametrization) that generalizes this phenomenon, first to singular 1-cubes in $\mathbf{R}^2$, then to singular 1-cubes in $\mathbf{R}^n$, then to singular k-cubes in $\mathbf{R}^n$.

2. For the standard 2-cube I_2, define the *oriented boundary* of I_2, denoted by ∂I_2, to be the "signed union" (we'll later call this a "1-chain") of the following sets:

   - $I_{(2,0)} = \{(t,0) \,|\, 0 \leq t \leq 1\}$ (with a + sign)
   - $I_{(1,1)} = \{(1,t) \,|\, 0 \leq t \leq 1\}$ (with a + sign)
   - $-I_{(2,1)} = -\{(t,1) \,|\, 0 \leq t \leq 1\}$ (the − sign is important!)
   - $-I_{(1,0)} = -\{(0,t) \,|\, 0 \leq t \leq 1\}$ (another minus sign!)

   (a) Explain why it's reasonable to think of $I_{(2,0)} + I_{(1,1)} - I_{(2,1)} - I_{(1,0)}$ as ∂I_2.

   (b) Suppose α is a 1-form on I_2. Prove that

   $$\int_{I_2} d\alpha = \int_{\partial I_2} \alpha,$$

   where the latter integral is defined via

   $$\int_{\partial I_2} \alpha = \int_{I_{(2,0)}} \alpha + \int_{I_{(1,1)}} \alpha - \int_{I_{(2,1)}} \alpha - \int_{I_{(1,0)}} \alpha.$$

   (If it's not clear what's going on, try an example or two – choose a specific 1-form that's easy but not trivial to integrate, to see how it works in a special case.)

   (c) Now let c be a singular 2-cube in $\mathbf{R}^2$. Explain how to define the oriented boundary ∂c of c, and then prove that, for a 1-form α on $\mathbf{R}^2$, we must have

   $$\int_c d\alpha = \int_{\partial c} \alpha.$$

   (This is sometimes called *Green's theorem in the plane.*)

   (d) Now let c be a singular 2-cube in $\mathbf{R}^n$. Explain how to define the oriented boundary ∂c of c, and then prove that, for a 1-form α on $\mathbf{R}^n$, we must have

   $$\int_c d\alpha = \int_{\partial c} \alpha.$$

   (In $\mathbf{R}^3$, this is what is usually called Stokes's theorem.)

   (e) Show that the integral of the 1-form $x\,dy$ around the boundary of a region in the plane is equal to the area of the region. Then verify this in the specific case of the unit disk.

3. The purpose of this exercise is to give a proof of the Poincaré lemma that is different from the one we'll do in class (and is more along the lines of the proof in the book), at least for a few special cases.

(a) Let U be a convex subset of the plane that contains the origin. Consider a 1-form α on U, suppose $\alpha = f(x, y)dx + g(x, y)dy$, and suppose that $d\alpha = 0$. The Poincaré lemma says that there is a function (0-form) $p(x, y)$ such that $dp = \alpha$.

Since

$$dp = \frac{\partial p}{\partial x}\, dx + \frac{\partial p}{\partial y}\, dy,$$

we need to find p so that

$$\frac{\partial p}{\partial x} = f, \qquad \frac{\partial p}{\partial y} = q.$$

To construct p, let $p(0, 0) = c$ for some constant c, and use the first of the two equations above to determine $p(x, 0)$ by setting

$$p(x, 0) = c + \int_0^x f(\xi)\, d\xi.$$

Then use the second of the two equations to determine $p(x, y)$ for $y \neq 0$ by setting

$$p(x, y) = p(x, 0) + \int_0^y g(x, \eta)\, d\eta.$$

Explain why this construction guarantees that

$$\frac{\partial p}{\partial y} = g$$

for all (x, y), and that

$$\frac{\partial p}{\partial x} = f$$

at least for points $(x, 0)$. Then, explain why the condition $d\alpha = 0$ guarantees that $\partial p/\partial x = f$ for all (x, y).

(b) Try and generalize this argument (integrating "one dimension at a time") to show that if ω is a 2-form in $\mathbf{R}^3$, such that $d\omega = 0$, then there is a 1-form β such that $d\beta = \omega$.