

Math 509
Assignment 8

Dr. DeTurck
Due Thursday, April 9

1. For an integer n and a real number $R > 0$, let $c_{R,n}: [0, 1] \rightarrow \mathbf{R}^2$ be the closed curve (singular 1-chain, or in fact singular 1-cycle)

$$c_{R,n}(t) = (R \cos nt, R \sin nt).$$

In fact, $c_{R,n}: [0, 1] \rightarrow \mathbf{R}^2 - \{0\}$. Intuitively, $c_{R,n}$ “winds” n times around the origin in \mathbf{R}^2 .

- (a) Show that for any integer n and any positive reals R_1, R_2 there is a 2-chain $c: [0, 1] \times [0, 1] \rightarrow \mathbf{R}^2 - \{0\}$ such that

$$\partial c = c_{R_1,n} - c_{R_2,n}.$$

- (b) Use Stokes’s theorem to show that if $R > 0$ and $n_1 \neq n_2$ then there is *no* 2-chain c such that

$$\partial c = c_{R,n_1} - c_{R,n_2}.$$

- (b) Now suppose \tilde{c} is a closed 1-cube (curve) in $\mathbf{R}^2 - \{0\}$, i.e., a map $\tilde{c}: [0, 1] \rightarrow \mathbf{R}^2 - \{0\}$ such that $\tilde{c}(0) = \tilde{c}(1)$. Show that there is a unique integer n and a 2-chain c such that

$$\partial c = \tilde{c} - c_{1,n}.$$

(One approach is to pick a line through the origin in \mathbf{R}^2 and a partition $0 = t_0 < t_1 < \dots < t_{K-1} < t_K = 1$ of $[0, 1]$ so fine that on $[t_i, t_{i+1}]$, the curve stays to one side or the other of the line.) The integer n is called the *winding number* of c around the origin.

2. Look at problem 21 on page 292 of the textbook (remember the textbook?). It’s like problem 1 above, except that Rudin calls “the index” what I have called “the winding number”. You don’t have to write anything down for 21, but do write up problems 22 and 23.
3. (Math 410 anyone?) We think of the set of complex numbers \mathbf{C} as the plane \mathbf{R}^2 and write $z = x + iy$ with $x, y \in \mathbf{R}$. So a complex function $f: \mathbf{C} \rightarrow \mathbf{C}$ can be thought of as a map from \mathbf{R}^2 to \mathbf{R}^2 , except that we write:

$$f(z) = f(x + iy) = u(x, y) + iv(x, y).$$

- (a) Certainly the simplest complex function is multiplication by a complex number $f(z) = \alpha z$ for $\alpha = a + ib \in \mathbf{C}$. Considered as a map from \mathbf{R}^2 to \mathbf{R}^2 , what is the derivative (Jacobian) of this map (as a 2-by-2 matrix)?

(b) One says that a mapping $F: (x, y) \rightarrow (u(x, y), v(x, y))$ is *holomorphic* if the derivative of F at each point is a 2-by-2 matrix of the form you found in part (a) (i.e., represents multiplication by a complex number). Show that this implies that u and v satisfy the *Cauchy-Riemann equations*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Since the derivative of F at $z = x + iy$ is a matrix that represents multiplication by a complex number, this number is denoted $f'(z)$.

(c) Practice: Show that $f(z) = z^2$, $f(z) = z^3$, $f(z) = z^p$ for p a positive integer are all holomorphic, and that their derivatives are what you would expect by simply using the calculus formulas. Also show that if $f(z) = e^z$, then $f'(z) = e^z$ as well. But show that $f(z) = \bar{z}$ is *not* holomorphic.

(d) Define $dz = dx + idy$, likewise $d(\alpha + i\beta) = d\alpha + id\beta$ and

$$\int_c \alpha + i\beta = \int_c \alpha + i \int_c \beta.$$

Show that $d(f \cdot dz) = 0$ if and only if f is holomorphic (i.e., f satisfies the Cauchy-Riemann equations).

(e) Use Stokes's theorem to prove the Cauchy Integral Theorem: if f is holomorphic on $A \subset \mathbf{C} = \mathbf{R}^2$, then

$$\int_c f \cdot dz = 0$$

for any closed curve c (i.e., singular 1-cube with $c(1) = c(0)$) such that $c = \partial\tilde{c}$ for some 2-chain \tilde{c} in A .

(f) Find the real and imaginary parts of the function $1/z$, and show that

$$\int_{c_{R,n}} \frac{1}{z} dz = 2\pi in,$$

where $c_{R,n}$ is as in problem 1.

(g) If f is holomorphic on \mathbf{C} , show that $g(z) = f(z)/z$ is holomorphic on $\mathbf{C} - \{0\}$. Then prove that for $R_1, R_2 > 0$,

$$\int_{c_{R_1,n}} g(z) dz = \int_{c_{R_2,n}} g(z) dz.$$

(h) Use problem 1, part (g) and a little analysis to prove Cauchy's Integral Theorem:

If c is a closed curve in $\mathbf{C} - \{0\}$ with winding number n around 0, and f is holomorphic, then

$$n \cdot f(0) = \frac{1}{2\pi i} \int_c \frac{f(z)}{z} dz$$

More generally, if z_0 is any complex number, and c is a closed curve in $\mathbf{C} - \{z_0\}$ with winding number n around z_0 , then

$$n \cdot f(z_0) = \frac{1}{2\pi i} \int_c \frac{f(z)}{z - z_0} dz.$$

... and so it goes.