**Math 509** **Dr. DeTurck**
**Assignment 9** **Due Tuesday, April 21**

1. Let $B_n \subset \mathbf{R}^n$ be the $n$-dimensional unit ball, and let $S^{n-1}$ be its boundary, the $n-1$-dimensional unit sphere.

(a) Give an example of an $n-1$-form $\alpha$ on $S^{n-1}$ such that

$$\int_{S^{n-1}} \alpha \neq 0.$$

(b) Explain why $\alpha$ must be a closed form on $S^{n-1}$, i.e., $d\alpha = 0$.

(c) In algebraic toplogy, if $X$ and $Y$ are topological spaces with $X \subset Y$ a *retract* (or retraction mapping) from $Y$ to $X$ is a continuous mapping $r\colon Y \to X$ such that $r(\mathbf{x}) = \mathbf{x}$ for every $\mathbf{x} \in X$. So if $Y$ is deformable to $X$, there is a retract from $Y$ to $X$. We (that is, you) are going to prove that there is no (differentiable) retract from $B_n$ to $S^{n-1}$.

- Suppose there is such a retract, $r\colon B_n \to S^{n-1}$. Prove that $r^*\alpha$ is a closed form on $B_n$, where $\alpha$ is the form in part (a).
- Use Stokes's theorem to show that this would imply that

$$\int_{S^{n-1}} \alpha = 0,$$

contradicting what you already know about $\alpha$. Therefore the differentiable retract $r$ cannot exist.

(d) Prove the differentiable version of the *Brower fixed-point theorem*: If $f\colon B_n \to B_n$ is a smooth map, then $f$ must have a fixed point, i.e., there must be a point $\mathbf{x} \in B_n$ satisfying $f(\mathbf{x}) = \mathbf{x}$. (*Hint*: If $f$ had no fixed point, then you could find a differentiable retract $r$ from $B_n$ to $S^{n-1}$ by defining $r(\mathbf{x})$ to be the point where the ray starting from $f(\mathbf{x})$ and going through $\mathbf{x}$ intersects $S^{n-1}$ – be careful to show that $r$ is differentiable and then apply part (c)).

(e) Prove that the system of equations

$$\begin{aligned} 2x + 3y &= 17 - \arctan(x^2 - e^y) \\ 5x + 4y &= \frac{\sin(x - y^3)}{3 + x^2 + 5y^4} \end{aligned}$$

has a solution.

2. What is the measure of the set of real numbers whose decimal expansion does not contain any 6's? (You might want to do the subset between 0 and 1 first.)

3. Exercises 2, 3 (the #3 on page 5), 4, 5, 8, 9 (the set $A$ here is $[0, \infty)$), 11, 12, 13.