Math 509 **Dr. DeTurck**
Practice Problems for Midterm 1 **Thursday, February 26**

1. Suppose $f\colon \mathbf{R}^n \to \mathbf{R}$ is a differentiable function and that $f(0,\ldots,0) = 0$. Prove that there exist functions $g_i\colon \mathbf{R}^n \to \mathbf{R}$ for $i = 1,\ldots,n$ such that
$$f(x) = \sum_{i=1}^{n} x_i g_i(\mathbf{x}).$$

2. Let $F\colon \mathbf{R}^2 \to \mathbf{R}$ be a differentiable function and $F(0,0) = 0$. Find conditions on F for which the implicit function theorem will guarantee a solution of the equation
$$F(F(x,y),y) = 0$$
for y as a function of x near $(0,0)$. How about for x as a function of y?

3. In neighborhoods of which points does the implicit function theorem guarantee you can solve
$$x^2 - yu = 0, \qquad xy + uv = 0$$
for (u,v) as functions of (x,y)? (By the way, what is the solution?)

4. Let $F\colon \mathbf{R}^n \to \mathbf{R}$ be a C^∞ function with the properties that $DF(0) = 0$ (all the partial derivatives of F are zero at the origin), and the second derivative $Q = D(DF)(0)$ is nonsingular. Show that there is a map $\mathbf{y}\colon \mathbf{R}^n \to \mathbf{R}^n$ (i.e., a change of coordinates) in a neighborhood of the origin such that $\mathbf{y}(0) = 0$, $D\mathbf{y}(0) = I$ and
$$F(\mathbf{y}(\mathbf{x})) = \frac{1}{2}\langle Q\mathbf{y}(\mathbf{x}), \mathbf{y}(\mathbf{x})\rangle$$
(so that with respect to the $\mathbf{y}$ coordinates, F is just a quadratic polynomial — this provides the classification of non-degenerate critical points and is called the *Morse lemma*).

   (Hint: Seek $\mathbf{y}(\mathbf{x})$ in the form $\mathbf{y}(\mathbf{x}) = R(\mathbf{x})\mathbf{x}$ where $R(\mathbf{x})$ is an invertible $n \times n$ matrix for each $\mathbf{x}$. The objective is to show that R can be chosen so that
$$F(\mathbf{x}) - F(0) = \frac{1}{2}\langle QR\mathbf{x}, R\mathbf{x}\rangle = \langle R^t QR\mathbf{x}, \mathbf{x}\rangle.$$
   But you can show that $F(\mathbf{x}) - F(0) = \frac{1}{2}\langle B(\mathbf{x})\mathbf{x}, \mathbf{x}\rangle$, where
$$B(\mathbf{x}) = 2\int_0^1 (1-t) F_{\mathbf{xx}}(t\mathbf{x})\, dt$$

Then use the implicit function theorem for the mapping from $n \times n$ matrices R to $n \times n$ *symmetric* matrices given by $R \mapsto R^t Q R$, to find $R(B)$ for B near Q such that $R(Q) = I$ and so that $R^t(B) Q R(B) = B$, then explain why this means you're done, using $R(\mathbf{x}) = R(B(\mathbf{x}))$.)

5. Suppose A is a rectangle (box) in $\mathbf{R}^n$ and f and g are integrable functions on A. Show that:

   (a) If $f(\mathbf{x}) \leq g(\mathbf{x})$ for all $\mathbf{x} \in A$, then

   $$\int_A f \leq \int_A g.$$

   (b) $|f(\mathbf{x})|$ is also integrable and

   $$\left| \int_A f \right| \leq \int_A |f|.$$

6. Suppose $f \colon \mathbf{R} \to \mathbf{R}$ is an increasing function on $[a, b]$. Show that the set $\{x \in [a, b] \mid f \text{ is discontinuous at } x\}$ has measure zero, hence f is integrable. (Hint: Consider the sets S_n of x for which $osc(f, x) \geq 1/n$.)