Math 509 **Dr. DeTurck**
Preliminary Problems for the Final Exam **Due Wednesday, May 6**

1. Let $U \subset \mathbf{R}^{n+1}$ be an open region such that the boundary $V = \partial U$ is a smooth n-dimensional hypersurface (if you want, just let U be the open ball $\{\mathbf{x} \in \mathbf{R}^{n+1} \,|\, \mathbf{x} \cdot \mathbf{x} < 1\}$, so that V is the unit sphere). Let f and g are two differentiable maps from V to V. We'll say that f and g are (smoothly) *homotopic* maps if there is a differentiable map $\varphi\colon V \times [0,1] \to V$ such that $\varphi(\mathbf{x}, 0) = f(\mathbf{x})$ and $\varphi(\mathbf{x}, 1) = g(\mathbf{x})$ for all $\mathbf{x} \in V$. (Notice that $\varphi(\mathbf{x}, t) \in V$ for all t, so the homotopy takes place entirely within V.)

   Now let ω be an n-form defined on V. If f and g are homotopic maps from V to V, show that
   $$\int_V f^*\omega = \int_V g^*\omega.$$
   So, for instance, if f is homotopic to the identity map, then
   $$\int_V f^*\omega = \int_V \omega$$
   and if f is homotopic to a constant map, then
   $$\int_V f^*\omega = 0.$$
   (Explain this last conclusion.)

2. Let $\{f_n\}$ be a sequence of measurable functions on $[0,1]$ such that, for each n, $|f_n(x)| < \infty$ for almost all x (i.e., the set $\{x \in [0,1] \,|\, |f(x)| = \infty\}$ has measure zero). Show that there exists a sequence $\{c_n\}$ of positive real numbers such that
   $$\lim_{n\to\infty} \frac{f_n(x)}{c_n} = 0$$
   for almost all x.

   (*Hint*: You get to choose c_n. Pick it so that the measure of the set where $|f_n(x)|/c_n > 1/n$ is small and apply the Borel-Cantelli lemma from the notes.)

3. Calculate
   $$\lim_{n\to\infty} \int_0^n \left(1 + \frac{x}{n}\right)^n e^{-2x}\, dx$$
   (be sure to justify any funny business with the limits!).

4. Suppose f is (Lebesgue) integrable on $[0, b]$ and set

$$g(x) = \int_x^b \frac{f(t)}{t}\, dt$$

for $0 < x \leq b$. Prove that g is integrable on $[0, b]$ and

$$\int_0^b g(x)\, dx = \int_0^b f(t)\, dt.$$

5. (This is not really an analysis problem, but is preliminary to something on the final.) Let x be any irrational number. Show that there exist infinitely many fractions p/q (with p and q relatively prime, i.e., the fraction is in lowest terms), such that

$$\left| x - \frac{p}{q} \right| \leq \frac{1}{q^2}.$$