UPPER BOUNDS FOR THE WRITHING OF KNOTS AND THE HELICITY OF VECTOR FIELDS

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TO JOAN BIRMAN ON HER 70TH BIRTHDAY

Abstract. The *writhing number* of a curve in Euclidean 3-space, introduced by Călugăreanu (1959-61) and named by Fuller (1971), is the standard measure of the extent to which the curve wraps and coils around itself; it has proved its importance for molecular biologists in the study of knotted duplex DNA and of the enzymes which affect it.

The *helicity* of a vector field defined on a domain in Euclidean 3-space, introduced by Woltjer (1958b) and named by Moffatt (1969), is the standard measure of the extent to which the field lines wrap and coil around one another; it plays important roles in fluid mechanics, magnetohydrodynamics, and plasma physics.

In this paper, we obtain rough upper bounds for the writhing number of a knot or link in terms of its length and thickness, and rough upper bounds for the helicity of a vector field in terms of its energy and the geometry of its domain. Then we describe the spectral methods which can be used to obtain sharp upper bounds for helicity and to find the vector fields which attain them.

Theorem A. *Let K be a smooth knot or link in 3-space, with length L and with an embedded tubular neighborhood of radius R. Then the writhing number $\mathrm{Wr}(K)$ of K is bounded by*

$$|\,\mathrm{Wr}(K)| < \frac{1}{4}\left(\frac{L}{R}\right)^{\frac{4}{3}}.$$

Theorem B. *Let V be a smooth vector field in 3-space, defined on the compact domain Ω with smooth boundary. Then the helicity $\mathrm{H}(V)$ of V is bounded by*

$$|\,\mathrm{H}(V)| \leq \mathrm{R}(\Omega)\,\mathrm{E}(V)\,,$$

where $\mathrm{R}(\Omega)$ is the radius of a round ball having the same volume as Ω and the energy of V is given by $\mathrm{E}(V) = \int_\Omega V \cdot V\, d(vol)$.

Theorem C. *The helicity of a unit vector field V defined on the compact domain Ω is bounded by*

$$|H(V)| < \frac{1}{2}\,\mathrm{vol}(\Omega)^{\frac{4}{3}}.$$

The *writhing number* $\mathrm{Wr}(K)$ of a smooth, simple, arc-length-parametrized curve K in 3-space is defined by the formula

$$\mathrm{Wr}(K) = \frac{1}{4\pi} \int_{K\times K} \left(\frac{dx}{ds} \times \frac{dy}{dt}\right) \cdot \frac{x-y}{|x-y|^3}\, ds\, dt\,,$$

while the *helicity* $\mathrm{H}(V)$ of a smooth vector field V on the domain Ω in 3-space is defined by the formula

$$\mathrm{H}(V) = \frac{1}{4\pi} \int_{\Omega\times\Omega} V(x) \times V(y) \cdot \frac{x-y}{|x-y|^3}\, \mathrm{d(vol)}_x\, \mathrm{d(vol)}_y\,,$$

where "smooth" for us always means of class C^∞.

Clearly, helicity for vector fields is the analogue of writhing number for knots. Both formulas are variants of Gauss' integral formula (1833) for the linking number of two disjoint closed space curves.

The upper bounds in Theorems A, B and C are not sharp. The proofs of Theorems A and C involve numerical integration: the actual constant obtained in Theorem A is roughly .233, which we overestimate as 1/4; that obtained in Theorem C is roughly .498, which we overestimate as 1/2. As for Theorem B, there exists, on a round ball Ω, a vector field V whose helicity is greater than one-fifth the asserted upper bound (see section 11), showing that this bound is of the right order of magnitude.

A bound similar to that expressed in Theorem A has been obtained independently, using different methods, by Buck and Simon (1999a,b); they show that

$$\mathrm{Wr}(K) \leq \frac{11}{4\pi} \left(\frac{L}{R}\right)^{\frac{4}{3}}.$$

Sharp upper bounds for helicity, and the spectral methods used to obtain them, are discussed in the second part of this paper.

We are grateful to De Witt Sumners and Craig Benham for their friendly guidance and counsel, for listening to earlier versions of this work, and for acquainting us with related works in the literature. We thank Jason Yunger for posing to us the question of upper bounds for the writhing number, and Eugenio Calabi for suggesting the Steiner symmetrization scheme in connection with one of our proofs of Theorem C.

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The paper is organized as follows:

I. ROUGH UPPER BOUNDS FOR WRITHING AND HELICITY

1. The writhing number of a curve.

Begin with two disjoint closed space curves, K and K', and with Gauss' integral formula for their linking number,

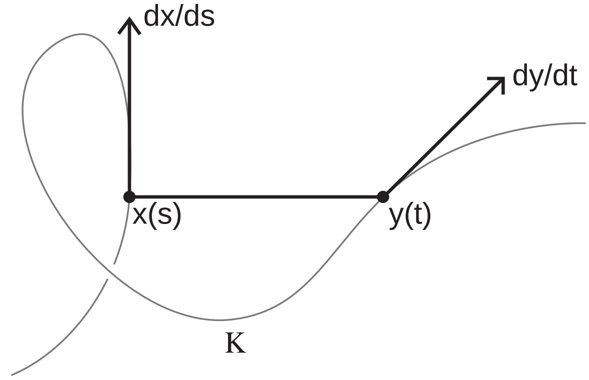
$$\mathrm{Lk}(K, K') = \frac{1}{4\pi} \int_{K \times K'} \left(\frac{dx}{ds} \times \frac{dy}{dt} \right) \cdot \frac{x - y}{|x - y|^3} \, ds \, dt \,. \tag{1.1}$$

The curves K and K' are assumed to be smooth and to be parametrized by arc–length.

In a series of three papers published in 1959-61, Georges Călugăreanu studied what happens to this integral when the two space curves K and K' come together and coalesce as one curve K, which we assume is simple.

At first glance, the integrand looks like it might blow up along the diagonal of $K \times K$, but a careful calculation shows that in fact the integrand approaches zero on the diagonal, and so the integral converges. Its value is the writhing number $\mathrm{Wr}(K)$ of K defined in the introduction:

$$\mathrm{Wr}(K) = \frac{1}{4\pi} \int_{K \times K} \left(\frac{dx}{ds} \times \frac{dy}{dt} \right) \cdot \frac{x - y}{|x - y|^3} \, ds \, dt \,. \tag{1.2}$$

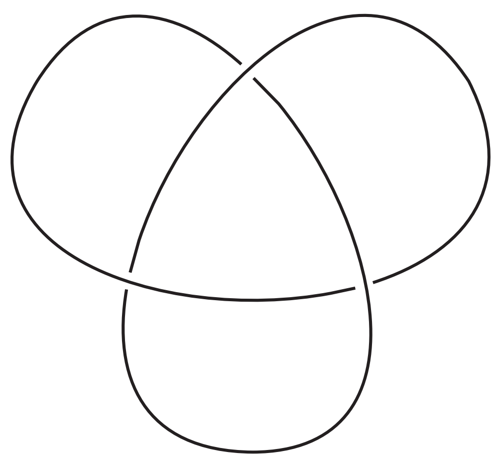


Here is a very useful fact, due to Fuller (1971):

The writhing number of a knot K is the average linking number of K with its slight perturbations in every possible direction:

$$\mathrm{Wr}(K) = \frac{1}{4\pi} \int_{W \in S^2} \mathrm{Lk}(K, K + \epsilon W)\, \mathrm{d(area)}\,. \tag{1.3}$$

This is helpful for getting a quick approximation to the writhing number of a knot which almost lies in a plane; in the example below, $Wr(K) \approx 3$.



For further information about writhing numbers in general, and especially about their use in molecular biology, we refer the reader to the papers of Pohl (1968), White (1969), Banchoff-White (1975), Fuller (1978), Bauer-Crick-White (1980), Wang (1982) and Sumners (1987, 1990, 1992).

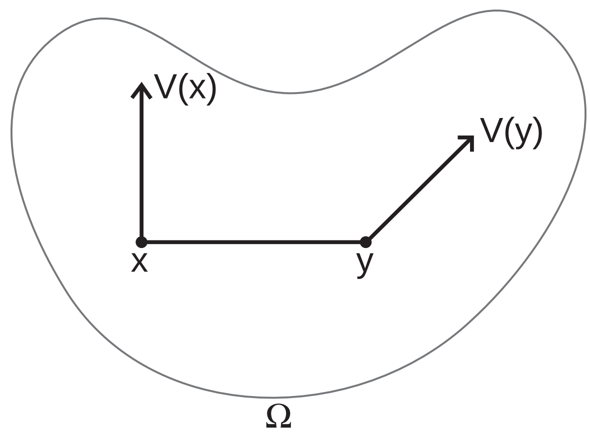
2. The helicity of a vector field.

Let Ω be a compact domain in 3-space with smooth boundary $\partial\Omega$; we allow both Ω and $\partial\Omega$ to be disconnected.

Let V be a smooth vector field, defined on the domain Ω.

Recall from the introduction that the helicity $\mathrm{H}(V)$ of the vector field V is defined by the formula

$$\mathrm{H}(V)=\frac{1}{4\pi}\int_{\Omega\times\Omega}V(x)\times V(y)\cdot\frac{x-y}{|x-y|^3}\,\mathrm{d(vol)}_x\,\mathrm{d(vol)}_y\,. \tag{2.1}$$



To help us understand this formula for helicity, think of V as a distribution of electric current, and use the Biot-Savart Law of electrodynamics (see Griffiths, 1989, pages 207-211) to compute its magnetic field:

$$\mathrm{BS}(V)(y)=\frac{1}{4\pi}\int_{\Omega}V(x)\times\frac{y-x}{|y-x|^3}\,\mathrm{d(vol)}_x\,. \tag{2.2}$$

Then the helicity of V can be expressed as an integrated dot product of V with its magnetic field $\mathrm{BS}(V)$:

$$\begin{aligned}\mathrm{H}(V)&=\frac{1}{4\pi}\int_{\Omega\times\Omega}V(x)\times V(y)\cdot\frac{x-y}{|x-y|^3}\,\mathrm{d(vol)}_x\,\mathrm{d(vol)}_y\\&=\int_{\Omega}V(y)\cdot\left[\frac{1}{4\pi}\int_{\Omega}V(x)\times\frac{y-x}{|y-x|^3}\,\mathrm{d(vol)}_x\right]\,\mathrm{d(vol)}_y\\&=\int_{\Omega}V(y)\cdot\mathrm{BS}(V)(y)\,\mathrm{d(vol)}_y\\&=\int_{\Omega}V\cdot BS(V)\,\mathrm{d(vol)}\,.\end{aligned}$$

We interpret this formula as follows.

Let $\mathrm{VF}(\Omega)$ denote the set of all smooth vector fields on Ω; then $\mathrm{VF}(\Omega)$ is itself an infinite-dimensional vector space. Define an inner product (called the L^2 *inner product*) on $\mathrm{VF}(\Omega)$ by the formula

$$\langle V,W\rangle=\int_{\Omega}V\cdot W\,\mathrm{d(vol)}\,. \tag{2.3}$$

Although the magnetic field $\mathrm{BS}(V)$ is well-defined throughout all of 3-space, we will restrict it to Ω; thus the Biot-Savart Law provides an operator

$$\mathrm{BS}(V) : \mathrm{VF}(\Omega) \to \mathrm{VF}(\Omega) .$$

Using the above inner product notation, our formula for the helicity of V can be written

$$\mathrm{H}(V) = \langle V, \mathrm{BS}(V) \rangle . \tag{2.4}$$

3. Boundedness of the Biot-Savart operator.

In this section, we will show that the Biot-Savart operator

$$\mathrm{BS}(V) : \mathrm{VF}(\Omega) \to \mathrm{VF}(\Omega)$$

is a bounded operator in the L^2 norm. That is, given the domain Ω, we will find a constant $C(\Omega)$ such that

$$|\,\mathrm{BS}(V)| \leq C(\Omega)|V|$$

for all smooth vector fields V on Ω, where

$$|\,\mathrm{BS}(V)|^2 = \langle \mathrm{BS}(V), \mathrm{BS}(V) \rangle$$

and

$$|V|^2 = \langle V, V \rangle = \mathrm{E}(V) ,$$

the *energy* of V.

Then, since $\mathrm{H}(V) = \langle V, \mathrm{BS}(V) \rangle$, it will follow that

$$|\,\mathrm{H}(V)| \leq C(\Omega)\,\mathrm{E}(V) .$$

This inequality, in the case that V is a divergence-free vector field, defined on and tangent to the boundary of a simply connected domain Ω, appears as Theorem 1.5, in Chapter III of Arnold and Khesin (1998), and goes back to Arnold (1974).

The proof that the Biot-Savart operator is bounded in the L^2 norm will follow along the lines of the usual Young's inequality proof for functions that convolution operators are bounded; see Folland (1995), page 9, or Zimmer (1990), Proposition B.3 on page 10. For clarity, we extract this proof as a lemma.

Lemma 3.1. *Let $\phi(x)$ be a scalar-valued function with the property that*

$$N_\Omega(\phi) = \max_y \int_\Omega |\phi(y-x)|\, d(vol)_x$$

is finite, where the maximum is taken over all points $y \in \mathbf{R}^3$. Then the operator $T_\phi : \mathrm{VF}(\Omega) \to \mathrm{VF}(\Omega)$ defined by

$$T_\phi(V)(y) = \int_\Omega V(x) \times \phi(y-x) \frac{y-x}{|y-x|}\, d(\mathrm{vol})_x$$

is a bounded map with respect to the L^2 norm, and furthermore,

$$|T_\phi(V)| \leq N_\Omega(\phi)\, |V| .$$

Proof. Fix $y \in \Omega$. Then, using the Cauchy-Schwarz inequality,

$$\begin{aligned}
|T_\phi(V)(y)| &\leq \int_\Omega |V(x)|\,|\phi(y-x)|\,\mathrm{d}(\mathrm{vol})_x \\
&= \int_\Omega |V(x)\{\phi(y-x)\}^{1/2}|\,|\{\phi(y-x)\}^{1/2}|\,\mathrm{d}(\mathrm{vol})_x \\
&\leq \left(\int_\Omega |V(x)|^2|\phi(y-x)|\,\mathrm{d}(\mathrm{vol})_x\right)^{1/2}\left(\int_\Omega |\phi(y-x)|\,\mathrm{d}(\mathrm{vol})_x\right)^{1/2} \\
&\leq (N_\Omega(\phi))^{1/2}\left(\int_\Omega |\phi(y-x)|\,|V(x)|^2\,\mathrm{d}(\mathrm{vol})_x\right)^{1/2}.
\end{aligned}$$

We square both sides, integrate with respect to y and use Fubini's theorem to get:

$$\begin{aligned}
\int_\Omega |T_\phi(V)(y)|^2\,\mathrm{d}(\mathrm{vol})_y &\leq N_\Omega(\phi)\int_\Omega\int_\Omega |\phi(y-x)|\,|V(x)|^2\,\mathrm{d}(\mathrm{vol})_x\,\mathrm{d}(\mathrm{vol})_y \\
&= N_\Omega(\phi)\int_\Omega |V(x)|^2\left(\int_\Omega |\phi(y-x)|\,\mathrm{d}(\mathrm{vol})_y\right)\mathrm{d}(\mathrm{vol})_x \\
&\leq N_\Omega(\phi)^2\int_\Omega |V(x)|^2\,\mathrm{d}(\mathrm{vol})_x.
\end{aligned}$$

Taking square roots, we get

$$|T_\phi(V)| \leq N_\Omega(\phi)\,|V|\,,$$

and conclude that T_ϕ is a bounded operator whose norm is at most $N_\Omega(\phi)$, as claimed. □

To apply this lemma to the Biot-Savart operator, we define the *optical size* of the domain Ω, written $\mathrm{OS}(\Omega)$, to be the number

$$\mathrm{OS}(\Omega) = \max_y \int_\Omega \frac{1}{|y-x|^2}\,\mathrm{d}(\mathrm{vol})_x\,, \tag{3.1}$$

where the maximum is taken over all points $y \in \mathbf{R}^3$. For a given value of y, this integral can be taken as a measure of the effort required to optically scan the domain Ω from the location y; the optical size of Ω is the maximum effort required to scan it from any location.

Then, in the language of Lemma 3.1,

$$\begin{aligned}
\mathrm{BS}(V)(y) &= \frac{1}{4\pi}\int_\Omega V(x) \times \frac{y-x}{|y-x|^3}\,\mathrm{d}(\mathrm{vol})_x. \\
&= T_{\phi_0}(V)(y)\,,
\end{aligned}$$

where

$$\phi_0(y-x) = \frac{1}{4\pi}\frac{1}{|y-x|^2}\,.$$

The lemma yields immediately that, for $V \in \mathrm{VF}(\Omega)$,

$$|\,\mathrm{BS}(V)| \leq \frac{1}{4\pi}\,\mathrm{OS}(\Omega)\,|V|\,, \tag{3.2}$$

and we conclude that $\mathrm{BS} : \mathrm{VF}(\Omega) \to \mathrm{VF}(\Omega)$ is a bounded operator.

4. Proof of Theorem B.

Theorem B. *Let V be a smooth vector field in 3-space, defined on the compact domain Ω with smooth boundary. Then the helicity $\mathrm{H}(V)$ of V is bounded by*

$$|\,\mathrm{H}(V)| \leq \mathrm{R}(\Omega)\,\mathrm{E}(V)\,,$$

where $\mathrm{R}(\Omega)$ is the radius of a round ball having the same volume as Ω and the energy of V is given by $\mathrm{E}(V) = \int_\Omega V \cdot V\, d(vol)$.

Proof. Theorem B will follow from the specific bound (3.2) for the norm of the Biot-Savart operator.

It is easy to see that the optical size of a domain Ω of given volume is maximized when Ω is a ball, with the point y chosen to be the center; the optical size of a ball is

$$\mathrm{OS}(\text{ball of radius } R) = 4\pi R\,.$$

Thus, if we define $\mathrm{R}(\Omega)$ to be the radius of a round ball having the same volume as Ω, then

$$\mathrm{OS}(\Omega) \leq 4\pi\,\mathrm{R}(\Omega)\,. \tag{4.1}$$

Hence

$$|BS(V)| \leq \frac{1}{4\pi}\,\mathrm{OS}(\Omega)\,|V| \leq \mathrm{R}(\Omega)\,|V|\,.$$

Then the helicity of V is bounded by

$$\begin{aligned} |\,\mathrm{H}(V)| &= |\,\langle V, \mathrm{BS}(V)\rangle\,| \leq |V|\,|\,\mathrm{BS}(V)| \\ &\leq \mathrm{R}(\Omega)\,|V|^2 = \mathrm{R}(\Omega)\,\mathrm{E}(V)\,, \end{aligned}$$

completing the proof of Theorem B. □

5. Proof of Theorem C.

A vector field on a given domain Ω maximizes its helicity for given energy by combining two strategies: selecting flow lines which coil well around one another, and distributing its energy so that the most crucial locations, in the "core" of the coiling, get the lion's share of the energy.

If we deprive a vector field of the second strategy by restricting it to be of unit length, then it will not be able to achieve as large helicity for given energy as before, and so we can expect to derive a more restrictive upper bound. When we convert this to an upper bound on the writhing number of a knot or link of given length and thickness by using the bridge theorem of Berger and Field to be described in the next section, we will obtain Theorem A.

We devote this section to proving

Theorem C. *The helicity of a unit vector field V defined on the compact domain Ω is bounded by*

$$|\,\mathrm{H}(V)| < \frac{1}{2}\,\mathrm{vol}(\Omega)^{\frac{4}{3}}.$$

Proof. Fix a location x within the domain Ω and define the helicity $\mathrm{H}(V,x)$ of V about x by the formula

$$\mathrm{H}(V,x) = \frac{1}{4\pi}\int_{\Omega} V(x)\times V(y)\cdot\frac{x-y}{|x-y|^3}\,\mathrm{d}(\mathrm{vol})_y\,, \tag{5.1}$$

so that the helicity $\mathrm{H}(V)$ of V is given by

$$\mathrm{H}(V) = \int_{\Omega}\mathrm{H}(V,x)\,\mathrm{d}(\mathrm{vol})_x\,. \tag{5.2}$$

Now let the fixed location be the origin in 3-space, and let $V(0) = \hat{z}$ be the unit vector pointing up along the z-axis. Let us seek both the domain Ω and the unit vector field V defined throughout Ω so as to maximize the helicity $\mathrm{H}(V,0)$ of V around the origin, subject only to the constraint that the volume of Ω is fixed in advance.

If the location y in Ω is given, then it is clear that choosing $V(y)$ to be the unit vector in the direction of $V(0)\times(y-0)$ will maximize the value of the integrand $V(0)\times V(y)\cdot(0-y)/|0-y|^3$ in the definition of $\mathrm{H}(V,0)$. Let $r = |y|$ and let θ denote the angle that the vector y makes with the horizontal plane, $-\pi/2 \le \theta \le \pi/2$. Then

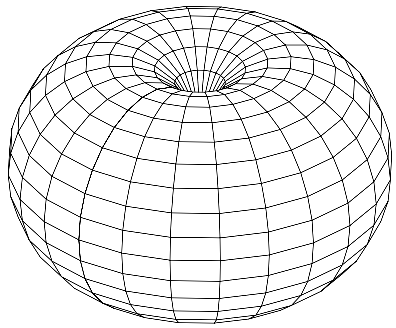
$$V(0)\times V(y)\cdot(0-y)/|0-y|^3 = (r\cos\theta)/r^3 = (\cos\theta)/r^2.$$

Hence each location y in Ω provides a contribution $(\cos\theta)/r^2$ to the integral which defines $\mathrm{H}(V,0)$.

Since we are seeking to maximize the value of $\mathrm{H}(V,0)$, and have the option of choosing the domain Ω subject only to fixed volume, we simply fill Ω with points in decreasing order of the value of $(\cos\theta)/r^2$, until we have the right volume. The resulting domain Ω will be a volume of revolution having the apple shape shown in the figure below, with boundary given by

$$(\cos\theta)/r^2 = \text{constant} = 1/k^2,$$

with the constant k selected so that Ω will have the preassigned volume.



Let us now evaluate $\mathrm{H}(V,0)$.

To begin, we use

$$\mathrm{d(vol)}_y = (2\pi r\cos\theta)(r\,dr\,d\theta)$$

to get

$$\begin{aligned}
\mathrm{H}(V,0) &= \frac{1}{4\pi}\int_\Omega (r^{-2}\cos\theta)(2\pi r\cos\theta)(r\,dr\,d\theta)\\
&= \frac{1}{2}\int_{\theta=-\pi/2}^{\pi/2}\left(\int_{r=0}^{k\sqrt{\cos\theta}} dr\right)\cos^2\theta\,d\theta\\
&= \frac{k}{2}\int_{-\pi/2}^{\pi/2}\cos^{5/2}\theta\,d\theta\\
&= \frac{kI}{2},
\end{aligned}$$

where

$$I = \int_{-\pi/2}^{\pi/2}\cos^{5/2}\theta\,d\theta \approx 1.4377$$

by numerical integration. We turn now to getting rid of the constant k by expressing it in terms of the volume of Ω.

$$\begin{aligned}
\mathrm{vol}(\Omega) &= \int_\Omega (2\pi r\cos\theta)(r\,dr\,d\theta)\\
&= 2\pi\int_{\theta=-\pi/2}^{\pi/2}\left(\int_{r=0}^{k\sqrt{\cos\theta}} r^2\,dr\right)\cos\theta\,d\theta\\
&= \frac{2\pi}{3}k^3\int_{-\pi/2}^{\pi/2}\cos^{5/2}\theta\,d\theta\\
&= \frac{2\pi}{3}k^3 I\,,
\end{aligned}$$

involving the same integral of $\cos^{5/2}\theta$ as before.

Thus

$$\mathrm{vol}(\Omega)^{1/3} = (2\pi/3)^{1/3} k I^{1/3}.$$

Now we substitute this into our formula for $\mathrm{H}(V,0)$ to get rid of the constant k, obtaining

$$\mathrm{H}(V,0) = (3/16\pi)^{1/3} I^{2/3}\,\mathrm{vol}(\Omega)^{1/3}.$$

Since the domain Ω and the unit vector field V in Ω were selected to maximize helicity about the origin, subject only to fixed volume, we can say in general that

$$|\,\mathrm{H}(V,x)| \le (3/16\pi)^{1/3} I^{2/3}\,\mathrm{vol}(\Omega)^{1/3}.$$

Since

$$\mathrm{H}(V) = \int_\Omega \mathrm{H}(V,x)\,\mathrm{d(vol)}_x$$

we can certainly conclude that

$$|\,\mathrm{H}(V)| \le (3/16\pi)^{1/3} I^{2/3}\,\mathrm{vol}(\Omega)^{1/3}\,\mathrm{vol}(\Omega)\,,$$

or

$$|\,\mathrm{H}(V)| \le (3/16\pi)^{1/3} I^{2/3}\,\mathrm{vol}(\Omega)^{4/3}.$$

Of course, this is not a careful overestimate, since if the domain Ω and the unit vector field V are chosen to maximize the helicity of V about one point of Ω, this will certainly not maximize the helicity about other points, nor the total helicity.

Inserting the numerical value of I into the last inequality above, we get

$$|\,\mathrm{H}(V)| < .498\,\mathrm{vol}(\Omega)^{4/3} < \frac{1}{2}\,\mathrm{vol}(\Omega)^{4/3},$$

completing the proof of Theorem C. □

The bound on the helicity of a unit vector field provided by Theorem C can also be obtained by an entirely different line of reasoning, as follows.

Begin with the definition of helicity,

$$\mathrm{H}(V) = \frac{1}{4\pi}\int_{\Omega\times\Omega} V(x)\times V(y)\cdot\frac{x-y}{|x-y|^3}\,\mathrm{d(vol)}_x\,\mathrm{d(vol)}_y\,,$$

Since V is a unit vector field, we certainly have

$$(5.3)\qquad |\,\mathrm{H}(V)| \le \frac{1}{4\pi}\int_{\Omega\times\Omega}\frac{1}{|x-y|^2}\,\mathrm{d(vol)}_x\,\mathrm{d(vol)}_y\,.$$

Suppose first that Ω is a unit ball, and explicitly calculate the above integral:

$$\int_{\Omega\times\Omega}\frac{1}{|x-y|^2}\,\mathrm{d(vol)}_x\,\mathrm{d(vol)}_y = 4\pi^2.$$

Thus

$$|\,\mathrm{H}(V)| \le \frac{1}{4\pi}4\pi^2 = \pi\,.$$

Now if Ω is a ball of any radius, the value of the above integral scales with the $4/3$ power of the volume of Ω, hence we can write

$$|\,\mathrm{H}(V)| \le C(\mathrm{vol}\,\Omega)^{4/3},$$

where $C = \pi(4\pi/3)^{-4/3} \approx .465$. So we certainly get

$$|\,\mathrm{H}(V)| < \frac{1}{2}(\mathrm{vol}\,\Omega)^{4/3},$$

for a unit vector field V on a round ball Ω.

The argument is now completed by showing that the integral

$$\int_{\Omega\times\Omega} \frac{1}{|x-y|^2}\,\mathrm{d}(\mathrm{vol})_x\,\mathrm{d}(\mathrm{vol})_y\,,$$

if compared among all domains Ω of given volume, is maximized when Ω is a ball. This is certainly plausible, because the integral represents the negative of the potential energy of a uniform distribution of mass over the domain Ω, under a gravitational attraction inversely proportional to the cube of the distance between particles. And so the integral is expected to be maximized (i.e., potential energy minimized) for a round "planet".

A proof that this is so can be given by the Steiner symmetrization method, following along the lines of the traditional proof of the isoperimetric theorem that round balls minimize surface area amongst domains of given volume. In fact, the proof in the present case is even easier than for the isoperimetric theorem. One first shows that symmetrizing a domain with respect to a plane will increase the value of our integral if the domain is not already symmetric with respect to that plane. If the domain is not a round ball, it must fail to be symmetric with respect to some plane. Hence no domain other than a round ball can possibly maximize our integral. The proof that the maximum exists then follows the classical argument, a beautiful description of which may be found on pages 248-255 of Blaschke's differential geometry book (1930).

Note how different in spirit this argument is from the one given before it. In the first argument, we carefully maximize the helicity of a unit vector field about one point. The slack in the argument comes when we use the value obtained to then bound the helicity about every point. By contrast, in the second argument, we immediately introduce slack by putting absolute value signs around the integrand, and then by bounding the absolute value of the triple product by the product of the absolute values of its factors. There is plenty of slack here, but after that, none at all. So it is rather curious that the two constants obtained,

(5.4) $$|\,\mathrm{H}(V)| < .498\,\mathrm{vol}(\Omega)^{4/3}$$

by the first argument, and

(5.5) $$|\,\mathrm{H}(V)| < .466\,\mathrm{vol}(\Omega)^{4/3}$$

by the second, are so close.

By contrast, if we apply Theorem B directly to a unit vector field, we only get

(5.6) $$|\,\mathrm{H}(V)| < .621\,\mathrm{vol}(\Omega)^{4/3}.$$

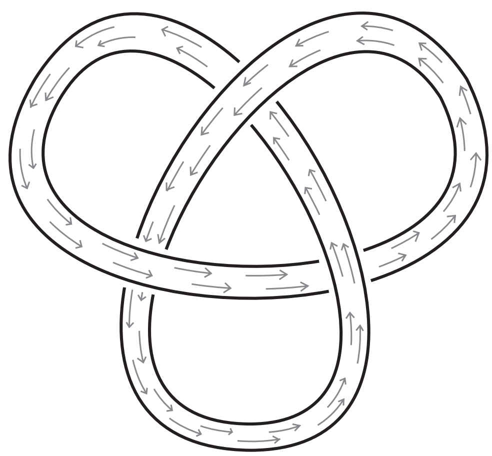
6. Helicity of vector fields and writhing of knots.

In this section we record a "bridge theorem", proved by Berger and Field (1984), which connects helicity of vector fields to writhing of knots and links, and which we will use in the next section to convert upper bounds on helicity into upper bounds on writhing. We also refer the reader to the two papers of Moffatt and Ricca (1992) for related results.

Proposition 6.1. [Berger and Field] *Let K be a smooth knot or link in 3-space and $\Omega = N(K, R)$ a tubular neighborhood of radius R about K. Let V be a vector field defined in Ω, orthogonal to the cross-sectional disks, with length depending only on distance from K. This makes V divergence-free and tangent to the boundary of Ω.*

Then the writhing number $\mathrm{Wr}(K)$ *of K and the helicity* $\mathrm{H}(V)$ *of the vector field V are related by the formula*

$$\mathrm{H}(V) = \mathrm{Flux}(V)^2\ \mathrm{Wr}(K)\,.$$



In the formula, $\mathrm{Flux}(V)$ denotes the flux of V through any of the cross-sectional disks D,

$$\mathrm{Flux}(V) = \int_D V \cdot n \ \mathrm{d(area)}\,,$$

where n is a unit normal vector field to D.

A key feature of this formula is that the helicity of V depends on the writhing number of K, but not any further on its geometry; in particular, such quantities as the curvature and torsion of K do not enter into the formula.

Berger and Field actually showed that the helicity $\mathrm{H}(V)$ is a sum of two terms: a "kink helicity", which is given by the right-hand side of the above formula, and a "twist helicity", which is easily shown in our case to be zero. Their proof assumes K is a knot, but it is straightforward to extend it to cover links.

We omit any further details and refer the reader to their paper.

7. Proof of Theorem A.

In this section, we use Berger and Field's bridge theorem between helicity and writhing number to convert upper bounds on helicity for unit vector fields into upper bounds for the writhing number of a knot or link.

Theorem A. *Let K be a smooth knot or link in 3-space, with length L and with an embedded tubular neighborhood of radius R. Then the writhing number* $\mathrm{Wr}(K)$ *of K is bounded by*

$$|\,\mathrm{Wr}(K)| < \frac{1}{4}\left(\frac{L}{R}\right)^{\frac{4}{3}}.$$

Proof. Let V be a unit vector field in Ω orthogonal to the disk fibres of the tubular neighborhood $\Omega = N(K, R)$.

From the previous section, we know that

$$\begin{aligned} \mathrm{H}(V) &= \mathrm{Flux}(V)^2\ \mathrm{Wr}(K) \\ &= (\pi R^2)^2\ \mathrm{Wr}(K)\,. \end{aligned}$$

We also have

$$\mathrm{vol}(\Omega) = \pi R^2 L\,,$$

independent of the shape of K, and hence, using the bound on the helicity of a unit vector field from Theorem C ,

$$|\,\mathrm{H}(V)| < \frac{1}{2}\,\mathrm{vol}(\Omega)^{4/3} = \frac{1}{2}(\pi R^2 L)^{4/3}.$$

Putting these together, we get

$$\begin{aligned} |\,\mathrm{Wr}(K)| &= |\,\mathrm{H}(V)|/\,\mathrm{Flux}(V)^2 = |\,\mathrm{H}(V)|/(\pi R^2)^2 \\ &< \frac{1}{2}\frac{(\pi R^2 L)^{4/3}}{(\pi R^2)^2} = \pi^{-2/3}(L/R)^{4/3} \\ &\approx .233(L/R)^{4/3} < \frac{1}{4}(L/R)^{4/3}, \end{aligned}$$

completing the proof of Theorem A. □

II. SHARP UPPER BOUNDS FOR HELICITY

In the following sections, we give a very brief overview of the methods used to find sharp upper bounds for the helicity of vector fields defined on a given domain Ω in 3-space, and include some pictures of the corresponding helicity- maximizing fields. Details can be found in our papers (1997a - c, 1998a - c).

8. The modified Biot-Savart operator.

As usual, Ω will denote a compact domain with smooth boundary in 3-space.

Let $\mathrm{K}(\Omega)$ denote the set of all smooth divergence-free vector fields defined on Ω and tangent to its boundary. These vector fields, sometimes called *fluid knots,* are prominent for several reasons:

1. They are the natural vector fields to study in a "fluid dynamics approach" to geometric knot theory.
2. They correspond to incompressible fluid flows inside a fixed container.
3. They are the vector fields most often studied in plasma physics.
4. The vector fields in this family which maximize helicity for given energy (equivalently, minimize energy for given helicity) provide models for stable force-free magnetic fields in gaseous nebulae and laboratory plasmas.
5. The search for these helicity-maximizing fields can be converted to the task of solving a system of partial differential equations.

The family $\mathrm{K}(\Omega)$ of fluid knots is a subspace of $\mathrm{VF}(\Omega)$; its orthogonal complement in the L^2 inner product is the subspace $\mathrm{G}(\Omega)$ of gradient vector fields:

$$\mathrm{VF}(\Omega) = \mathrm{K}(\Omega) \oplus \mathrm{G}(\Omega);$$

see our paper (1997b) on the Hodge Decomposition Theorem.

Start with a vector field V in $\mathrm{K}(\Omega)$, and compute its magnetic field, $\mathrm{BS}(V)$. Restrict $\mathrm{BS}(V)$ to Ω and subtract a gradient vector field so as to keep it divergence-free while making it tangent to $\partial\Omega$. The resulting vector field $\mathrm{BS}'(V)$ can be viewed as the orthogonal projection of $\mathrm{BS}(V)$ back into $\mathrm{K}(\Omega)$.

Then

$$\mathrm{BS}' : \mathrm{K}(\Omega) \to \mathrm{K}(\Omega)$$

is the *modified Biot-Savart operator.*

For vector fields V in $K(\Omega)$, we have

$$\mathrm{H}(V) = \langle V, \mathrm{BS}'(V) \rangle \,; \tag{8.1}$$

this follows from the corresponding formula $\mathrm{H}(V) = \langle V, \mathrm{BS}(V) \rangle$, since $\mathrm{BS}(V)$ and $\mathrm{BS}'(V)$ differ by a gradient vector field, which is orthogonal in the inner product structure of $\mathrm{VF}(\Omega)$ to any vector field V in $\mathrm{K}(\Omega)$.

9. Spectral methods.

From now on, we will focus on vector fields which are divergence-free and tangent to the boundary of their domain, that is, on the subspace $\mathrm{K}(\Omega)$ of $\mathrm{VF}(\Omega)$, and on the modified Biot-Savart operator $\mathrm{BS}' : \mathrm{K}(\Omega) \to \mathrm{K}(\Omega)$. We refer the reader to our paper (1997c) for the proof of the following result (we already established boundedness in section 3 of the present paper).

Theorem 9.1. *The modified Biot-Savart operator* BS' *is a bounded operator, and hence extends to a bounded operator on the* L^2 *completion of its domain; there it is both compact and self-adjoint.*

The spectral theorem then promises that the extended BS' behaves like a real self-adjoint matrix: its domain, the L^2 completion of $\mathrm{K}(\Omega)$, admits an orthonormal basis of eigenfields, in terms of which the operator is "diagonalizable". The eigenfields corresponding to the eigenvalues $\lambda(\Omega)$ of maximum absolute value are the vector fields with maximum absolute value of helicity for given energy, and we obtain the sharp upper bound

$$|\,\mathrm{H}(V)| \leq |\lambda(\Omega)|\,\mathrm{E}(V) \tag{9.1}$$

for all V in $\mathrm{K}(\Omega)$.

This approach to helicity was initiated by Arnold (1974) in his study of the asymptotic Hopf invariant on closed orientable 3-manifolds.

10. Connection with the curl operator.

If the vector field V is divergence-free and tangent to the boundary of its domain Ω, then it is a standard fact of physics, embodied in one of Maxwell's equations, that

$$\nabla \times \mathrm{BS}(V) = V; \tag{10.1}$$

see Griffiths (1989), pp. 215-217, and our paper (1997c).

Since $\mathrm{BS}(V)$ and $\mathrm{BS}'(V)$ differ by a gradient vector field, we also have

$$\nabla \times \mathrm{BS}'(V) = V\,. \tag{10.2}$$

If V is an eigenfield of BS',

$$\mathrm{BS}'(V) = \lambda V\,,$$

then

$$\nabla \times V = \frac{1}{\lambda} V\,. \tag{10.3}$$

Thus the eigenvalue problem for BS' can be converted to an eigenvalue problem for curl on the image of BS', which means to a system of partial differential equations. Even though we extended BS' to the L^2 completion of $\mathrm{K}(\Omega)$ in order to apply the spectral theorem, the eigenfields are smooth vector fields in $\mathrm{K}(\Omega)$; this follows, thanks to elliptic regularity, because on divergence-free vector fields, the square of the curl is the negative of the Laplacian. Hence these vector fields can be (and are) discovered by solving the above system of partial differential equations.

It turns out (see our paper 1997c) that the equation $\nabla \times \mathrm{BS}(V) = V$ holds if and only if V is divergence-free and tangent to the boundary of Ω. These two restrictions on V are therefore necessary for the conversion of the eigenvalue problem for the Biot-Savart operator to a system of partial differential equations, as described above.

In the following section we describe two instances in which this system of partial differential equations can be completely solved.

11. Explicit computation of helicity-maximizing vector fields.

We solve $\nabla \times V = (1/\lambda)V$ on the flat solid torus $D^2(a) \times S^1$, where $D^2(a)$ is a disk of radius a and S^1 a circle of any length. Although this is not a subdomain of 3-space, the solution here is so clear-cut and instructive as to be irresistable. See our paper (1997a) for details.

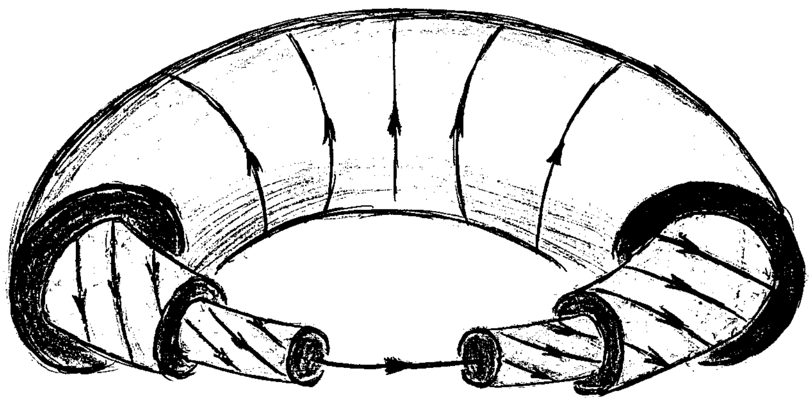
The eigenvalue of BS′ of largest absolute value is

$$\lambda(D^2(a) \times S^1) = a/2.405..., \tag{11.1}$$

where the denominator is the first positive zero of the Bessel function J_0. The corresponding eigenfield, discovered by Lundquist in 1951 in his study of force-free magnetic fields, is

$$V = J_1(r/\lambda)\hat{\phi} + J_0(r/\lambda)\hat{z}\,, \tag{11.2}$$

expressed in terms of cylindrical coordinates (r, ϕ, z) and the Bessel functions J_0 and J_1.



It follows that if V is any vector field in $\mathrm{K}(D^2(a) \times S^1)$, then

$$|\,\mathrm{H}(V)| \leq (a/2.405...)\,\mathrm{E}(V)\,, \tag{11.3}$$

with equality for the above eigenfield V .

We solve $\nabla \times V = (1/\lambda)V$ on the round ball $B^3(a)$ of radius a in terms of spherical Bessel functions. See our paper (1998b), written with Mikhail Teytel, for details.

The eigenvalue of BS′ of largest absolute value is

$$\lambda(B^3(a)) = a/4.4934...; \tag{11.4}$$

the denominator is the first positive zero of $(\sin x)/x - \cos x$.

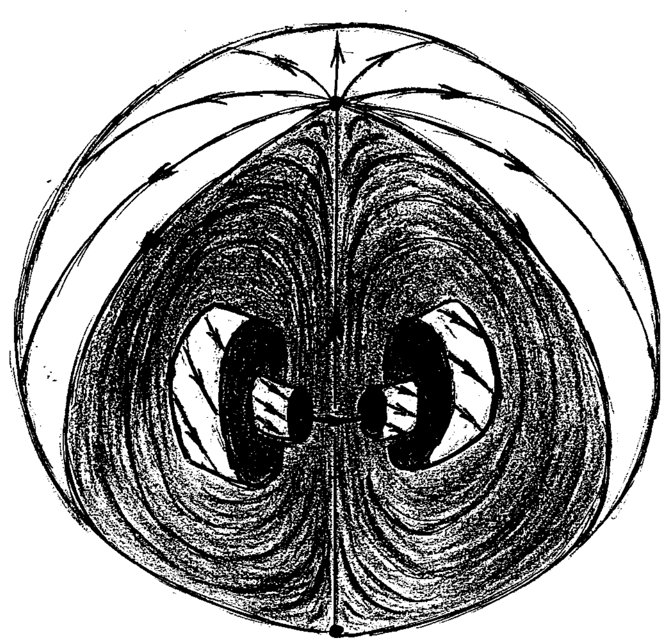
The corresponding eigenfield is Woltjer's model (1958a) for the magnetic field in the Crab Nebula (also known as the *spheromak field* in plasma physics), described below in spherical coordinates (r, θ, ϕ) on a ball of radius $a = 1$:

$$V(r, \theta, \phi) = u(r, \theta)\hat{r} + v(r, \theta)\hat{\theta} + w(r, \theta)\hat{\phi}\,,$$

where

$$
\begin{aligned}
u(r,\theta) &= \frac{2\lambda}{r^2}\left(\frac{\sin(r/\lambda)}{r/\lambda} - \cos(r/\lambda)\right)\cos\theta \\
v(r,\theta) &= -\frac{1}{r}\left(\frac{\cos(r/\lambda)}{r/\lambda} - \frac{\sin(r/\lambda)}{(r/\lambda)^2} + \sin(r/\lambda)\right)\sin\theta \\
w(r,\theta) &= \frac{1}{r}\left(\frac{\sin(r/\lambda)}{r/\lambda} - \cos(r/\lambda)\right)\sin\theta\,.
\end{aligned}
$$

Note that the value $\lambda = 1/4.4934...$ makes both $u(r,\theta)$ and $w(r,\theta)$ vanish when $r = 1$, that is, at the boundary of the ball. As a consequence, the vector field V is tangent to the boundary of the ball, and directed there along the meridians of longitude.



It follows that if V is any vector field in $\mathrm{K}(B^3(a))$, then

$$
|\,\mathrm{H}(V)| \leq (a/4.4934...)\,\mathrm{E}(V)\,, \tag{11.5}
$$

with equality for the above eigenfield V.

Compare this with the earlier rough upper bound from Theorem B:

$$
|H(V)| \leq a\,\mathrm{E}(V)\,.
$$

12. The isoperimetric problem and the search for optimal domains.

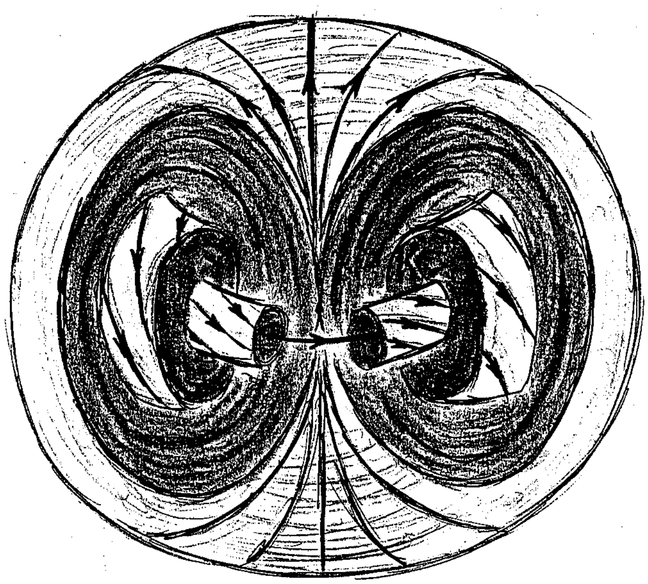
Up to this point, we have been holding the domain Ω fixed and trying to maximize helicity among all divergence-free vector fields of given energy, defined on and tangent to the boundary of Ω.

We now let the domain Ω itself vary, while fixing its volume, and so come to the *isoperimetric problem* in this setting:

Maximize helicity among all divergence-free vector fields of given energy, defined on and tangent to the boundary of all domains of given volume in 3-space.

Equivalently, we seek to maximize the largest absolute value of an eigenvalue of the modified Biot-Savart operator BS' among all domains of given volume in 3-space.

This problem is treated in detail in our paper (1998a), written jointly with Mikhail Teytel. There we derive first variation formulas for these extreme eigenvalues of the modified Biot-Savart operator, and apply them to learn that the spheromak field on the round ball, described in the preceding section, is not the absolute helicity-maximizing field with given energy on a domain of given volume. Instead, the ball on which it is defined can be dimpled in at the poles and expanded out at the equator to further increase the helicity, while preserving both the energy of the field and the volume of the domain. Our numerical computations, guided by these first variation formulas, suggest that this volume-preserving, energy-preserving, helicity-increasing deformation of domain and field converges to a singular domain, in which the north and south poles have been pressed together at the center, along with a corresponding singular field, as shown below.



We expect the topology of the helicity-maximizing field to be essentially the same on the optimizing domain, the flat solid torus and the round ball, with the field in each case tangent to a family of nested tori with a single core curve.

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