Math 660 Review

This aims to be a (very) quick overview of the entirety of MA 660, including the test, modulo some scalar product facts.

1 Scalar (Inner) product spaces

In this section, we defined a scalar product and talked about some of its basic properties. In particular, many of the properties inner products have carry over to scalar products, though often the proofs were more involved.

Definition 1.1. A scalar product $<,>$ on a vector space V is a bilinear paring satisfying:

1. $< u, v > = < v, u >$
2. $< u, u >$ is nondegenerate, i.e., $\forall v \neq 0 \in V, \exists u \in V$ such that $< u, v >\neq 0$.

Definition 1.2. To each scalar product $<,>$, we associate a quadratic form $q : V \to V$ defined by $q(v) =< v, v >$. It is possible to recover $<,>$ from q by using the polarization identity: $< u, v >= \frac{1}{2}q(v + u) - q(v) - q(u)$.

Definition 1.3. An inner product $<,>$ is a scalar product in which

3. $u \neq 0 \Rightarrow < u, u >$ ¿ 0.

Notice 3⇒2.

Definition 1.4. Let (V,$<,>$) be a scalar product space. The index, ν, is defined to be the dimension of the largest subspace of V on which $<,>$ is negative definite. In particular, an inner product has index 0.

Definition 1.5. If (V,$<,>_V$) and (W,$<,>_W$ are inner product spaces, then a linear map $f : V \to W$ is an isometry if $\forall u, v \in V, < u, v >_V =< f(u), f(v) >_W$. This is equivalent to the statement: $\forall v \in V, q_V(v) = q_W(f(v))$, where q_V and q_W are the associate quadratic forms.

Theorem 1.1. *If (V,$<,>_V$) and (W,$<,>_W$) are two scalar product spaces, then they are isometric iff dim(V) = dim(W) and the index of $<,>_V$ = the index of $<,>_W$.*

2 (Semi-)Riemannian metrics

Basically, from here on we assumed that almost all scalar products were in fact inner products, and we shifted towards doing more classical Riemannian geometry. After this section, scalar products were almost not mentioned at all.

Definition 2.1. Let M be a manifold. A Riemannian metric tensor g on M is a symmetric (2,0)-tensor which is positive definite. In other words, for each $p \in M$, we associate an innerproduct g_p on T_pM which varies smoothly. The pair (M, g), or just M when feeling lazy is called a Riemannian manifold. I will use g and $<,>$ more or less interchangeably.

Definition 2.2. If $U \subseteq M$ is any chart on M, then we can express g in local coordinates as follows. Let $x^1, ..., x^n$ be the coordinates on U and let $\frac{\partial}{\partial x^i}$ be the associated coordinate vector fields. Let $g_{ij} = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$. It then follows that for arbitary vector fields Y and Z on U, $g(Y, Z) = g_{ij}Y^iZ^j$ where $Y = Y^i\frac{\partial}{\partial x^i}$ and $Z = Z^j\frac{\partial}{\partial x^j}$. In then follows that we can write $g = g_{ij}dx^i \otimes dx^j$, where dx^i is the dual basis to $\frac{\partial}{\partial x^i}$.

Definition 2.3. If $N \subset M$ is a submanifold and M has a Riemannian metric g on it, we can put a metric on N as follows: For each $p \in N$ and for each $v, w \in T_pN$, let $g|_N(v, w) = g(v, w)$, where we think of $T_pN \subseteq T_pM$.

More generally, if $f : N \to M$ is a smooth immersion map, and M is a Riemannian manifold with metric g, then we can put a Riemannian metric on N as follows: for $n \in N$ and for $u, v \in TpN$, let $g_N(u, v) = g_M(f_*u, f_*v)$. In other words, $g_N = f^*g_M$.

Definition 2.4. Let (N, g_N) and (M, g_M) be Riemannian manifolds with $f : M \to N$. f is said to be an isometry if f is a diffeomorphism and $f^*g_M = g_N$.

Note 2.1. It is clear that the composition of isometries is an isometry and that the inverse of an isometry is an isometry. Thus, Iso(M,g) is a group. Further, it's in fact a Lie group (though I"'m not sure how to prove this). Finally, if M is compact, then so is Iso(M,g) - but I'm fairly certain this is a nontrivial fact.

Note 2.2. The isometry group of the sphere S^n is $O(n + 1)$, which has dimension $\frac{n(n+1)}{2}$. $SO(n + 1)$, the connected component of $O(n + 1)$, is also the collection of orientation preserving isometries of S^n. $SO(2) = S^1$ (easy). $SO(3) = RP^3$ (consider $f : S^3 \to SO(3)$, where $f(u) = (p \to upu^{-1})$. This is a double covering map). $SO(4) = S^3 \times RP^3$ (consider $f : S^3 \times S^3 \to SO(4)$

where $f(u, v) = (p \rightarrow upv^{-1})$. Again, f is a double cover. Note that "‘equality"’ is to interpreted as lie group isomorphic in the case of $SO(2)$ and $SO(3)$, but only as diffeomorphism for $SO(4)$. For n bigger than 4, $SO(n)$ isn't diffeomorphic to anything familiar.

Definition 2.5. Let $c : [a, b] \rightarrow M$ be a smooth curve in a Riemannian manifold M. Then the length of c, $L(c)$ is defined by $L(c) = \int_a^b |\frac{dc}{dt}|dt$.

Theorem 2.1. *Every (paracompact) manifold M admits a metric.*

Proof. Choose a partition of unity λ_i subordinate to any locally finite covering of M by coordinate charts U_i. In any chart U_i, let g_i be defined by the quadratic from $ds^2 = \sum(dx^j)^2$. Let $g = \sum \lambda_i g_i$. Then g is a riemannian metric on M. ☐

Note 2.3. If we repace "‘Riemannian metric"’ by "‘metric of previously chosen index ν, then the theorem is false. As a counterexample, S^2 doesn't admit a metric of index 1. The proof of this fact is simple: Assuming g is an index one metric , at each point p there is a 1-d subspace W of T_pM so that $g|_W = 0$. Choosing a spanning vector of W at each point gives rise to a smooth nonvanishing vector field on S^2, which is impossible.

3 Connections

The motivation for a connection is as follows: Tangent vectors give us the tools neccesary to differentiat functions, but suppose we want to differentiate vectors with respect to vectors? More specifically, a geodesic intuitively shouldn't deviate it all - it's acceleration should be 0. But acceleration is a second order differential operator, so we'll need to be able to differentiate the derivative of the curve, i.e., we need to be able to differentiate vector fields. The fundamental problem with this is that a derivative is almost always a limit of a difference equation, but invariably, the difference will be between two vectors living in different vector spaces. So, we must find some way to "‘transport"’ a vector from one vector space to another. Lie derivatives give one way, but there is only a single Lie derivative, so it's hard to make sense of a lie derivative being "‘compatible"’ with a metric. Connections give another way, with a natural sense of being "‘compatible"’ with a metric. That said, connections are of interest in their own right - it can be proved, for example, that there are connections which aren't "‘compatible"’ with any metric. Also, notice that almost none of this chapter requires a metric.

Definition 3.1. Let $VF(M)$ denote the set of smooth vector fields on M. $\nabla : VF(M) \times VF(M) \to VF(M)$, with $\nabla(X,Y) = \nabla_X Y$, is called a connection if for any $f, g \in C^\infty(M)$ and for any $X, Y, Z \in VF(M)$

1. $\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z$ (∇ is tensorial in the first coordinate).
2. $\nabla_X Y + Z = \nabla_X Y + \nabla_X Z$
3. $\nabla_X fY = f\nabla_X Y + X(f)Y$. (Leibnitz rule) (Notice that since X(c) = 0 for all constants c, this implies linearity in the second coordinate over $\mathbb{R}$.

Note 3.1. The fact that ∇ is tensorial in the first coordinate implies that the value of $\nabla_X Y$ at a point p only depends on the value of X at the point p. Further, since the second coordinate behaves as a Liebnitz rule, the value of $\nabla_X Y$ at a point p only depends on the value of Y in an arbitrarily small neighborhood of p.

Definition 3.2. A connection ∇ is called symmetric if $\nabla_X Y - \nabla_Y X = [X, Y]$.

An immediate consequence of the existence of a connection is the following:

Theorem 3.1. *Given any smooth curve $c : I \to M$, then there is a unique map taking the smooth vector fields on the curve to smooth vector fields on the curve, denoted by $V \to \frac{DV}{dt}$ with the following properties:*

1. $\frac{D(V+W)}{dt} = \frac{DV}{dt} = \frac{DW}{dt}$.

2. $\frac{DgV}{dt} = g\frac{DV}{dt} + \frac{dg}{dt}V$.
3. If V *can be extended to M,* $\frac{DV}{dt} = \nabla_{\frac{dc}{dt}}V$.
In other words we can now differentiate vector fields along curves.

Definition 3.3. A vector field V along a curve c is parallel if $\frac{DV}{dt} = 0$.

It is a consequence of ODEs that given any vector at c(0), we can parallel translate it in a unique way to any point c(t).

Definition 3.4. c is called a geodesic if $\frac{D\frac{dc}{dt}}{dt} = 0$, i.e., if $\frac{dc}{dt}$ is parallel along c. (Notice we can talk about geodesics WITHOUT talking about metrics!)

At this point, we'll now throw in metrics.

Definition 3.5. A connection ∇ on M is compatible with a metric g if for any smooth curve $c : [0,1]$, if V and W are parallel on c, then $g(V(0), W(0)) = g(V(t), W(t))$ for all $t \in [0,1]$. This is equivalent to the following: $\frac{d}{dt} < V, W >=< \frac{dV}{dt}, W > + < V, \frac{dW}{dt} >$, and also that $U < V, W >=< \nabla_U V, W > + < V, \nabla_U W >$.

And now, the theorem which makes everything work:

Theorem 3.2. *(Levi-Civita)Given a metric g, there exists a unique symmetric connection* ∇ *which is compatible with g. If "'symmetric"' is dropped, one still has existence, but one loses uniqueness.*

Note 3.2. During the course of the proof, the following two formulae pop out:
1. In any chart U, $\Gamma^m_{ij} = \frac{1}{2}(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ki}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k})g^{km}$, where $\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j} = \Gamma^m_{ij}\frac{\partial}{\partial x^m}$.
2. $2 < \nabla_U V, W >= U < V, W > + V < W, U > - W < U, V > - < U, [V, W] > + < V, [W, U] > + < W, [U, V] >$.

4 Geodesics

We had a definition of geodesic before we had a definition of metric. Now that we have a connection compatible with the metric, we'll find geodesics have many of the properties we expect of them, at least if we're willing to stick to small enough distances.

Note 4.1. We take a quick pause to describe the geodesic equation in coordinates. If Γ_{ij}^k are the Christoffel symbols in some chart, then the geodesic equation is $\frac{d^2x^k}{dt^2} + \Gamma_{ij}^k \frac{dx^i}{dt}\frac{dx^j}{dt} = 0$.

Theorem 4.1. *For any* $p \in M$ $\exists U^{open} \subseteq M$ *and a* δ *and* $\epsilon > 0$ *such that* $\exists \gamma : (-\delta, \delta) \times TU_\epsilon \to M$ *such that* $t \to \gamma(t, p, v)$ *is the unique geodesic passing through* p *at time 0 with tangent vector* v. $TU_\epsilon = \{(q, v) \in TU : |v| < \epsilon\}$. *This is proved by appealing to fundamental thoerems from the theory of first order ODEs. Further, it's clear that by shrinking* ϵ *by a factor c, we can increase* δ *by a factor of c. Thus, by choosing* ϵ *sufficiently small, we may assume* $\delta = 2$.

Definition 4.1. The map $exp : TU_\epsilon \to M$ defined by $exp(q, v) = \gamma(1, q, v)$ is called the exponential map.

Theorem 4.2. $\exists \epsilon > 0$ *such that* $exp|_{B_\epsilon(0)}$ *is a diffeomorphism (proven by an easy use of the inverse function theorem).*

From here, we aim to prove that geodesics locally minimize lengths. This is done in several steps

Theorem 4.3. *(Gauss Lemma) Let* $(s, t) \to X(s, t)$ *be a smooth survae in* M *with the feature that the s-coordinates are geodesics, then* $\frac{D}{ds}\frac{\partial X}{\partial t} = \frac{D}{dt}\frac{\partial X}{\partial s}$. *Further, this implies that* $\frac{\partial}{\partial s} < \frac{\partial X}{\partial s}, \frac{\partial X}{\partial t} >= 0$.

Theorem 4.4. *If* $S_r \subseteq B_r(0) \subseteq T_pM$ *is the sphere and* $S_p = exp_p(S_r)$, *then the geodesics through* p *are orthogonal to* S_p. *Under these conditions, we can put "'polar"' coordinates on the normal neighborhood. In these coordinates, the metric looks like* $ds^2 = dr^2 + dS_p^2$ *where* dS_p^2 *is the induced metric on the sphere. From here, it's obvious that geodesic rays are unique shortest paths to their endpoints.*

Theorem 4.5. *The converse, that a path which minimizes distance and is paramaterized by arc length is a geodesic - first, it's clear that it's a broken geodesic (around each point, put a normal ball... continue with picture), then use even finer (i.e., totally normal) neighborhoods to remove corners.*

5 Invariant metrics on Lie groups

Definition 5.1. Given a Lie group G, a metric g is called left invariant if $L_h^* g = g$ for all $h \in G$, where L_h is the map which sends p to hp. Similarly, G is right invariant if $R_h^* g = g$, and bi invariant if it's both left and right invariant. It is easy to make left invariant metrics: start with any metric g_e on $T_e G$ and define $g_p = L_{p^{-1}}^* g_e$. Bi-invariant metrics are a bit harder to come by, but exist on all compact lie groups.

Theorem 5.1. *The following are equivalent:*

1. A metric is bi-invariant

2. At the identity, the metric is invariant under C_{g*} where C_g denotes conjugation by g.

3. $< [X, U], V > + < U, [X, V] >= 0$ for any left invariant vector fields X,U, and V.

Theorem 5.2. *If X is right invariant vector field on G, where the metric on g is left invariant, then the flow of X through the identity is a 1-parameter subgroup. If G has a bi-invariant metric, then the geodesics through the identity and the 1-parameter subgroups coincide.*

Proof. Let V be a left invariant vector field and let γ be it's flow through the identity. We must show $\frac{DV}{dt} = 0$. Since both $V|_\gamma$ and $\frac{d\gamma}{dt} = V$ are extendible, this is equivalent to $\nabla_V V = 0$.

Now, using something from above, we have $2 < \nabla_U V, W >= U < V, W > +V < W, U > -W < U, V > - < U, [V, W] > + < V, [W, U] > + < W, [U, V] >$. If we let $U = V$ and W also be left-invariant vector fields, then the inner products are constant, and hence their derivaties are 0 and we're left with $< \nabla_V V, W >=< V, [W, V] >$. From (3) above, this implies that $< V, [W, V] >= 0$ and hence $< \nabla_V V, W >= 0$. Since W is arbitrary, we get $\nabla_V V = 0$. Finally, since there is a one-parameter subgroup through the identity in each direction, this gives us all the possible geodesics, so the converse is proven. □

(There will be much more to say about Lie Groups once we have the notion of curvature)

6 Curvature

Definition 6.1. The curvature is a map from $VF(M) \times VF(M) \times VF(M) \rightarrow VF(M)$, denoted by $R(X,Y)Z$ given by $R(X,Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z$. Note that some authors use the negative of this.

Note 6.1. 1. It's immediately obvious that $R(X,Y)Z = -R(Y,X)Z$.

2. The term $\nabla_{[X,Y]}Z$ is there in order to turn R into a tensor. This is important because it means the value of $R(X,Y)Z$ at a point p depends only on the values of X, Y, and Z at p.

3. Plugging in coordinate vector fields in local coordinates yields $R(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})\frac{\partial}{\partial x^k} = (\nabla_{x^j}\nabla_{x^i} - \nabla_{x^i}\nabla_{x^j})\frac{\partial}{\partial x^k}$. Thus, in some sense, R measures the noncommutativity of mixed partial derivatives.

Definition 6.2. The Riemann curvature tensor is the map $R : VF(M) \times VF(M) \times VF(M) \times VF(M) \rightarrow \mathbb{R}$, given by $(X,Y,Z,T) \rightarrow < R(X,Y)Z, T >$.

Theorem 6.1. *(Bianchi identities) We'll use* (X,Y,Z,T) *to denote* $< R(X,Y)Z, T >$, *simply because it's much easier to write*

1.$(X,Y,Z,T) + (Y,Z,X,T) + (Z,X,Y,T) = 0$

2.$(X,Y,Z,T) = -(Y,X,Z,T)$.

3.$(X,Y,Z,T) = -(X,Y,T,Z)$.

4.$(X,Y,Z,T) = (Z,T,X,Y)$.

Definition 6.3. Let $u, v \in T_pM$ be independent. The sectional curvature of the two-plane σ spanned by u and v is defined by $K(\sigma) = \frac{<R(u,v)u,v>}{|u|^2|v|^2 - <u,v>}$. Geometrically, we think of using the exponential map on an open subset of σ to get a 2-dimensional submanifold of M. Then $K(\sigma)$ reports the Gaussian curvature of the submanifold.

Theorem 6.2. *R at a point p is completely determined by K at a point p. This is a purely algebraic fact.*

Definition 6.4. M has constant section curvature at p if $K(\sigma)$ is independent of σ. M has constant sectional curvature if it has constant sectional curvature at each point p. A Theroem of Schur (and an excercise in Do Carmo) states that if $dimM \geq 3$, and M has constant sectional curvature at each point p, then the curvature cannot vary from point to point.

Theorem 6.3. *M has constant sectional curvature K at p iff* $(X,Y,Z,T)_p = K(|X|^2|Y|^2 - < X,Y >)_p$. *The proof is obvious - it's almost exactly a definition.*

Definition 6.5. The Ricci curvature at a point p is a map from $VF(M) \rightarrow \mathbb{R}$. It can be defined in 3 equivalent ways:

1. The contraction of the curvature tensor

2. $Ric(v) = \frac{1}{n-1} \sum K(v, z_i)$, where z_i is an O.N. basis for T_pM which is orthogonal to v.

3. $Ric(v) = \frac{1}{area(S^{n-2})} \int_{S^{n-2}} K(v, y)dy$

Definition 6.6. The scalar curvature at p can be defined similarly to the above, but the most common formula is $Scal(p) = \frac{1}{n} \sum Ric(z_i) = \frac{1}{n(n-1)} \sum_{i<k} K(z_i, z_j)$ where z_i is an O.N. basis for T_pM.

Note 6.2. Sectional curvature is a stronger notion than ricci curvature, which is in turn stronger than scalar curvature. As we've seen in algebraic topolgy, we have the following (difficult) theorem due to Kazdan:

Theorem 6.4. *Let $f \in C^\infty(M)$ and assume $\exists p \in M$ such that $f(p) < 0$. Then there is a metric on M such that f is the scalar curvature.*

. The situation for positive scalar curvature is almost as loose - most "'normal"' manifolds admit a metric of positive scalar curvature, but there are some known exceptions.

On of my favorite things about curvature is that there is a lot of interplay between metrics of strictly positive, strictly negative, non-positive, non-negative curvature and the topology of M.

7 Curvature of (bi-)invariant metrics on lie groups

We have already seen some interesting facts about the relationship between geodesics through the origin and 1-parameter subgroups on a Lie group G with bi-invariant metric. There are also some VERY nice relationships concerning with curvature

A calculation shows the following:

Theorem 7.1. *(O'Neill's formula) If G has a bi-invariant metric, then*

1.$\nabla_X Y = 1/2[X, Y]$.

2.$R(X, Y)Z = 1/2[[X, Y], Z]$.

3.*For X, and Y orthonormal, and* $\sigma = span\{X, Y\}$, $K(\sigma) = 1/4||[X, Y]||^2$. *In particular, G has non-negative curvature.*

That said, we have the following:

Theorem 7.2. *Assume G is a compact Lie group with left invariant metric. Further, suppose G has positive curvature. Then G is diffeomorphic to* S^3 *or* $RP^3 = SO(3)$.

Proof. If G actually has a bi-invariant metric, then one appeals to the classification of semisimple lie algebras to show for any lie algebra L other that $so(3), \exists u, v \in L$ such that $[u, v] = 0$. To handle the more general left-invariant case in even dimensions isn't too bad. One first proves that every Killing field on a positively curved compact manifold has a 0. Then one shows that right invariant vector fields are killing fields. So, G clearly has a nonzero right invariant vector field. But a right invariant vector field is a killing field, so has a 0. But if a right invariant vector field has a 0, it's the 0 field, a contradiction. The odd dimensional requires a (much more difficult) theorem by Takagi: G cpt with positive curvature and X, and Y killing fields implies either $[X, Y] = 0$ or they're dependent at some point. One then appeals to the classification of semisimple lie algebras again. ☐

8 Jacobi Fields

The arise very naturally as fields of variations of geodesics. Intuitively, they will measure the rate of spread of geodesics. This, in term, will lead to information about critical points of the exponential map.

Definition 8.1. Let $f = f(t,s)$ be a parameterized surface in M. Assume the t-coordinates are geodesics. Let $V = V(t,s)$ be a vector field along $f(t,s)$. Since the lie bracket of coordinate vector fields is 0, we have $\frac{D}{ds}\frac{D}{dt}V - \frac{D}{dt}\frac{D}{ds}V = R(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s})V$. If we know specialize to $V = \frac{\partial f}{\partial t}$, then the following holds: $0 = \frac{D}{dt}\frac{\partial f}{\partial t} = R(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s})\frac{\partial f}{\partial t} + \frac{D}{dt}\frac{D}{ds}\frac{\partial f}{\partial t} = R(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s})\frac{\partial f}{\partial t} + \frac{D}{dt}\frac{D}{dt}\frac{\partial f}{\partial s}$. Now, let $\gamma(t) = f(t,0)$ and let $J(t)\frac{\partial f(t,0)}{\partial s}$, then we get that $\frac{D^2 J}{dt^2} + R(\gamma'(t), J(t))\gamma'(t) = 0$. This final equation is known as the Jacobi equation. Any solution to it is known as a Jacobi field.

Theorem 8.1. *A Jacobi field J(t) is determined by J(0) and J’(0) (=*$\frac{DJ}{dt}$*). Further, For any two vectors v and w in* $T_{\gamma(0)}M$*, exists a (unique) Jacobi field with* $J(0) = v$ *and* $J'(0) = w$*. The proof is an appeal to the theory of second order ODEs.*

Note 8.1. We can actually do much better than this. If we assume $v = 0$ in the above, we have an explicit construction of J as follows; Let $V(s)$ be any smooth map in T_pM with $V(0) = v$ and $V'(0) = W$, where we have identified $T_0(T_pM) = T_pM$. Let $f(t,s) = exp_p(tV(s))$. Let $J(t) = \frac{\partial f(t,0)}{\partial s}$. Then $J(t)$ is the desired Jacobi field.

Jacobi fields are important because they measure the spread of geodesics

Theorem 8.2. *Under the conditions above, with* w, v *orthonormal, we have* $|J(t)|^2 = t^2 - 1/3K(\sigma)t^4 +$*remainder.*

To see this more explicitely, assuming we have constant curvature we can solve the Jacobi equations explicitly to yield one of the following 3 cases (where W(t) is the parellel translate of $w = J'(0)$):

1.$J(t) = \frac{\sin(t\sqrt{K})W(t)}{\sqrt{K}}$ if $K > 0$.
2.$J(t) = tW(t)$ if $K = 0$.
3.$J(t) = \frac{\sinh(t\sqrt{-K})W(t)}{\sqrt{-K}}$ if $K < 0$.

This confirms our guess that in positve curvature, geodesics should come closer to each other, while in negative curvature they spread apart.

Definition 8.2. if $\gamma : [0,a] \to M$ is a geodesic, then $\gamma(t_0)$ is said to be a conjugate point if there is a Jacobi field J such that J is not identically 0 and $J(t_0) = 0$. The multiplicity of a conjugate point is the number of linearly independent Jacobi fields which vanish at $\gamma(t_0)$. This number is less than or equal to n-1 since there can be at most n fields which vanish there, but the field $t\gamma(t)$ can easily be shown to be a Jacobi field which never vanishes after $t = 0$.

Theorem 8.3. *let* q*=*$\gamma(t_0)$ *and let* $v = t_0\gamma'(0)$*. Then*

1. q is conjugate to $\gamma(0)$ *iff v is a critical point of the exponential map at p and further*

2. The multiplicity of q is equal to the dimension of the kernel of $d_v exp_p$

In particular, if γ has no conjugate points and if exp_p is defined on all of T_pM, then exp_p is a local diffeomorphism between R^n and $exp(T_pM)$ (We will see later that under these circumstances, exp_p is surjective and is in fact a covering map.)

Further, it's somewhat clear (and a simple calculation verifies) that if $K \leq 0$ for all points p in a manifold M, then Jacobi fields grow larger as t increases. In particular, there are no conjugate points.

9 Hyperbolic plane

We are already quite familiar with a standard constant positive curvature manifold (the sphere), and also with a flat (curvature 0) manifold (R^n), but we're not familiar with anythign with constant negative sectional curvature. In this section, we cover a simply connected manifold with constant negative curvature (eventually, we'll prove it's the only one). It's diffeomorphic to R^n but has a somewhat strange metric on it.

We have 2 main models of the 2 dimensional version - the upper half plane with $ds^2 = \frac{1}{y^2}(dx^2 + dy^2)$, and the unit disc (no boundary) with $ds^2 = \frac{4(dx^2+dy^2)}{1-x^2-y^2}$.

Theorem 9.1. *1. Each of these has constant curvature -1 (a simple calculation verifies this)*

2. They are, in fact, isometric, by the isometry $f(z) = \frac{z-i}{z+i}$, where we think of the upper half plane as $\Im(z) > 0$

This isometries in each model are (draw on board - use obvious symmetries and whichever model best helps you to see them). The isometry group acts transitively on points and transitively on tangent spaces. In otherwords, it's as large as possible.

In this section we also encountered:

Theorem 9.2. *(Minding) Any two surfaces with the same (Gaussian) curvature are locally isometric.*

To be honest, I'm not sure of the significance of local isometries.

Another way to see the isometries of hyperbolic space is to notice that the collection of all mӧbius transforms ($z \longrightarrow \frac{az+b}{cz+d}$ with ad-bc=1) is the full isometry group (technically, we need to mod out by $a = d = -1$).

10 Complete manifolds

Complete manifolds rock. They are a generalization of compact manifolds which behave well in Riemannian geometry.

Definition 10.1. A Riemannian manifold M is complete if for each $p \in M$, exp_p is defined on all of T_pM.

Definition 10.2. We can define a distance function $d : M \times M \to \mathbb{R}$ by $d(a, b) = GLB\{\text{paths joining a and b}\}$, assuming M is connected. Then

1. M becomes a metric space
2. The topology on M induced from d is the same as the topology M originally had
3. If there is a minimizing geodesic joining a and b, $d(a, b) = L(\gamma)$.

We are now ready for one of the most useful theorems in all of Riemannian geometry

Theorem 10.1. *(Hopf-Rinow) The following are equivalent:*

1. *There is a single $p \in M$ such that exp_p is defined on all of T_pM.*
2. *Closed and bounded subsets of M are compact.*
3. *(M,d) is complete as a metric space. (Cauchy sequences converge).*
4. *M is geodesically complete (every geodesic runs for all time)*
5. *exp_p is defined on all of T_pM for all $p \in M$.*
6. *There is a sequence of compact subsets of M, $K_1 \subseteq K_2 \subseteq ... \subseteq M$ such that if $q_n \in M - K_n$, then $d(p, q_n) \to \infty$.*

Further, any of the above imply that for any two points p and q, there is a length minimizing geodesic joining p and q.

Theorem 10.2. *(corollaries) A compact manifold is complete. A closed subset of a complete manifold is complete.*

We are now ready for:

Theorem 10.3. *(Hadamard) If M is complete with $K \leq 0$, then M is covered by R^n. Thus, if M is simply connected, M is diffeomorphic to R^n.*

Proof. Let $p \in M$. As mentioned above, the nonpositive curvature assumption implies exp_p is a local diffeomorphism. We then have the following:

Lemma 10.4. *If N is complete and we have a local diffeomorphism $f : N \to M$ such that for any $p \in N$, $|d_pf(v)| \geq |v|$, then f is a covering map. The proof of this is to show that N has the path lifting property.*

With this lemma in hand, we proceed as follows. Let g the metric on M. We DEFINE g_p, a metric on T_pM by $g_p = exp_p^* g$. Then T_pM is complete because the geodesics through 0 correspond to geodesics in M, which run for all time. Then exp_p becomes a local isometry, so the hypothesis of the lemma are satisfied. Thus, exp_p is a covering map.

□

11 Curvature of Surfaces in 3-space

The purpose of this section is (at least) two-fold. First, a general review of the differential geometry of surfaces is nice in its own right. Second, we seek to build intuition before moving onto higher dimensions/more complex things.

Definition 11.1. Let S be an orientable surface in 3-space, and let $X : U \to V \subseteq S$ be a chart. We let $N(p) = \frac{X_u \times X_v}{|X_u \times X_v|}$, where we choose an orientation ahead of time to make everything make sense. If we think of $N : S \to S^2$, then we call this the Gauss map. Since being normal to S and being normal the sphere coincide, we can identify T_pS with $T_{N(p)}S^2$. Thus, we can think of $d_pN : T_pS \to T_pS$. It is straightforward to show that $d_pN(X_u) = N_u(p)$ and likewise for X_v.

We also saw some of the more classical theorems

Theorem 11.1. *Given $p \in M$, let k_1 denote the smallest curvature obtained on a curve through p, and let k_2 denote the largest. Let $K = k_1k_2$ and let $H = 1/2(k_1 + k_2)$. Then $K = \det d_pN$ and $H = -1/2$ trace d_pN.*

Definition 11.2. The second fundamental form, $II(\alpha') = - < d_pN(\alpha'), \alpha' >= - < N_uu' + N_vv', X_uu' + X_vv' >$

12 Isometric Immersions

We now turn to a more general idea. Let $f : M^m \to N^n$ be an immersion, (so $n \geq m$). If $f^* g_N = g_M$, then we call f and isometric immersion. What we hope to accomplish is as follows: "'our goal is to study the relationship between the geometry of M and N..."'. For convenience, we will assume f is an inclusion map, which we can get away with since it locally is. Notice, then, that for each $p \in M$, $T_pN = T_pM \oplus (T_pM)^{\perp}$.

Definition 12.1. Let ∇^M denote the covariant derivative on M and let ∇^N denote the covariant derivative of N. For X, Y vector fields on M, let $B(X,Y) = \nabla^N_{\overline{X}} \overline{Y} - \nabla^M_X Y$, where $\overline{X}$ denotes any local extension of X to N, and similarly for $\overline{Y}$. Then, B is well defined, B is symmetric in X and Y, B is linear over $C^\infty(M)$, and thus the value of B only depends on X and Y at p. B is called the second fundamental form. Notice that $B_p : T_pM \times T_pM \to (T_pM)^{\perp}$. We can change this to a real values form by choosing an $\eta \in (T_pM)^{\perp}$, and defining $H_\eta(X,Y) =< B(X,Y), \eta >$. From H_η, we get a linear self-adjoint map in the usual way, $S_\eta : T_pM \to T_pM$, where $< S_\eta(X), Y >= H_\eta(X,Y)$. S_η can be computed as $S_\eta(X) = -(\nabla^N_X N)^{\perp}$, where N is a local normal vector field along M with $N(p) = \eta$.

With this crazy definition in place, we have the following

Theorem 12.1. *(Gauss Lemma) Let $X, Y \in T_pM$. Then $K_M(X,Y) - K_N(X,Y) =< B(X,X), B(Y,Y) > - < B(X,Y), B(X,Y) >$.*

Definition 12.2. $M^m \subseteq N^n$ is said to be totally geodesic if the second fundamental form $B : T_pM \times T_pM \to (T_pM)\perp$ is identically 0 at all points p. This is equivalent to the statement that geodesics of M are geodesics of N.

Theorem 12.2. *(Cartan). If for each $p \in M$ and each σ a two plane of T_pM, there is a totally geodesic submanifold tangent to σ, then M has constant sectional curvature. (We didn't prove this)*

13 Spaces of constant curvature

Our goal is to partially classify all complete spaces of constant curvature up to isometry. One fact that hurts us is the following: there exist 2 surfaces $X(u,v)$ and $Y(u,v)$ of nonconstant curvature such that curvature of $X(u,v)$ = curvature of $Y(u,v)$ at (u,v), but are not even locally isometric. In other words, in general, knowing the curvature does NOT allow you to recover the metric.

So, we still want a (local) isometry given that we start with the same curvatures, so here's how we proceed:

Definition 13.1. Let $f : U \subseteq M \to U' \subset N$ be of the form $f = exp_{p'} \circ \phi \circ (exp_p)^{-1}$, where $\phi : T_pM \to T_{p'}N$ is an isometry. Then f is called a pre-isometry. f clearly preserves distances in the 'radial' direction, but in general will NOT preseverse distance in the 'angular' directions.

However, we have

Theorem 13.1. *if for all $q \in U$, and all $X, Y, Z, T \in T_pM$, we have $< R_M(X,Y)Z, T >=< R_N(\phi(X), \phi(Y))\phi(Z), \phi(T) >$, then f is an isometry (on U).*

Proof. The idea is to use Jacobi fields to measure the spread of radial geodesics, allowing us to get a handle on measuring distance in the "'angular"' directions. The fact that the curvatures agree will force the "'spread"' to stay the same, making distance measurements in "'angular"' directions the same. ☐

With this in hand, we're ready for:

Theorem 13.2. *Let M be a complete manifold with constant sectional curvature K = (resp, K=0, resp K= -1). Then M is covered by a sphere (resp, R^n, resp, hyberbolic space). In particular, if M is simply connected, it's either isometric to S^n, or R^n with the boring metric or hyperbolic metric.*

Proof. First, we can rescale a metric and assume K = 1, 0, or -1. For $K \leq 0$, the proofs isn't so bad - it's essentially hadamard's thereom with Cartan's lemma added to it. For $K = 1$, a bit of care is needed, but it essentially boils down to using the above procedure twice. ☐

Definition 13.2. A space form M is a complete connected space of constant curvature. The above theorem implies they are all quotients of one of the 3 models spaces (quotients via actions of $\Pi_1(M)$). Thus, space forms can be classified by classifying all "'nice"' subgroups of the isometry groups of the model spaces.

One relatively easy theorem is the following:

Theorem 13.3. *if M is even dimensional and $K = 1$, them M is isometric to either S^n or RP^n*

Proof. One classfies all subgroups of $SO(2n+1)$ (the isometry group of the 2n-sphere), which act freely easily using basic linear algebra. The only one is $\frac{\mathbb{Z}}{2\mathbb{Z}} = \{Id, -Id\}$. $\square$

In odd dimensions, things are MUCH more complicated. As an example, there is a free action of $\frac{\mathbb{Z}}{p\mathbb{Z}}$ on any odd dimensional sphere for any natural number p

14 Variations of Energy

If I hand you a curve $c : I \to M$, a very natural question to ask is "'what happens to the length as I perturb it in this direction"'. Variations of energy are sort of the tools developed to answer a question like this.

Definition 14.1. A variation of c is a continuous mapping $f : (-\epsilon, \epsilon) \times I \to M$ such that

1.$f(0, t) = c(t)$.

2. there is a subdivision of I so that $f(s, t)$ is smooth on that subdivision.

If $f(s, 0) = c(0)$ (and similarly for the endpoint), then f is called a fixed endpoint variation.

The variation field of the variation is defined by $V(t) = \frac{\partial f(0,t)}{\partial s}$. The basic theory of ODEs tells us that for any piecewise smooth vector field $V(t)$, we can construct a variation having $V(t)$ as its variation field.

Definition 14.2. $L : (-\epsilon, \epsilon)$, with $L(s) = \int_I |\frac{\partial f(s,t)}{\partial t}| dt$, the length function. However, it's actually more convenient to work with the energy function $E : (-\epsilon, \epsilon)$, where $E(s) = \int_I |\frac{\partial f(s,t)}{\partial t}|^2 dt$.

Theorem 14.1. *if $I = [0, a]$, then $L(c)^2 \leq aE(c)$, with equality iff c is paramaterized proportional to arc length. If γ is a minimizing geodesic joining $\gamma(0) = p$ to $\gamma(a) = q$, then for any other curve c with $c(0) = p$ and $c(a) = q$, we have $E(\gamma) \leq E(c)$, with equality holding iff c is also a minimizing geodesic.*

Theorem 14.2. *(First variation) Let $c : [0, a] \to M$ is a curve with $f : (-\epsilon, \epsilon) \times [0, a] \to M$ a variation. If the interior corners occur at $t_1 < t_2 < ... < t_k$, then we have $1/2E'(0) = -\int_0^a < V(t), \frac{Dc'}{dt} > dt - \sum_i < V(t_i), c'(t_i^+) - c'(t_i^-) > - < V(0), c'(0) > + < V(a), c'(a) >$.*

Notice that if c is in fact a geodesic and if the variation is a fixed point variation, then this theorem tells us that $E'(0) = 0$. In fact, this is equivalent to c being a geodesic. Thus, if we're looking at geodesics, the first bit of interesting information comes from E''.

Theorem 14.3. *Same hypothesis as in the previous theorem, except this time $c = \gamma$ is a geodesic and f is a fixed endpoint variation. Then $1/2E''(0) = -\int_0^a < V, V'' + R(\gamma', V)\gamma' > dt = \int_0^a < V', V' > - < R(\gamma', V)\gamma', V >$.*

This calculation has AMAZING consequences.

Theorem 14.4. *(Bonnet - Myers)If M has ricci curvature $> 1/r^2 > 0$ Then any minimizing geodesic has lenght $\leq \pi r$. Thus, if M is complete, $M = exp_p(\overline{B_{\pi r}(0)})$, the image of a compact set, and hence is compact. Since ricci curvature is the average of the sectional curvatures, if M has sectional curvatures bounded above by $1/r^2$, then the theorem applies.*

Proof. (here's the proof for the sectional curvature case) Assume for a contradiction that there are points of distance greater than πr. Let γ be a geodesic realizing this. Let $V(t) = sin(\pi t/a)U(t)$, where $U(t)$ is orthogonal to γ' and is parallel. Then the formula for 2nd variation quickly shows that $1/2E''(0) < 0$, which implies for small variation, we shrink energy and thus length. The more general version with ricci curvature involves a more careful managing, but the idea is the same. □

As a quick corolary, we have

Theorem 14.5. *Let M be complete with Riccie curvatures $\geq \delta > 0$. Then the universal covering space is compact and thus, $\Pi_1(M)$ is finite.*

We also have the following (difficult?) theorem, due to Cheng and Shiohama.

Theorem 14.6. *If M is complete with Ricci curvature $\geq 1/r^2 > 0$, then we know from Bonnet-Myers that the diameter $\leq \pi r$. If in fact it's equal, then M is isometric the the usual sphere.*

The following two theorems are proved in Do Carmo (via variations), one of which is also proved in the notes:

Theorem 14.7. *(Synge-Weinstein) Let M^n be compact. Let f be an isometry of M. If n is even, assume f preserves orientation. If n is odd, assume f reverses orientation. Then f has a fixed point.*

Proof. if f doesn't have a fixed point, let $p \in M$ minimize $d(p, f(p))$, and let γ be a minimizing geodesic between p and $f(p)$. Use a nice variation to minimize things further, contradicting your choice of γ. □

Theorem 14.8. *(Synge)if M is compact with positive curvature, then*

1. If n is even, then $\Pi_1(M) = \frac{\mathbb{Z}}{2\mathbb{Z}}$ or 0, depending on whether or not M is orientable or not.

2. If n is odd, then M is orientable.

Proof. If M has even dimension and M is orientable, then let $\overline{M}$ denote the universal cover, with covering metric. Then any isometry of $\overline{M}$ will preserve orientation since M was orientable, so has a fixed point. Since an isometry is a deck transformation, a fixed point means it's the identity. Thus, the deck group $= \Pi_1(M)$ is trivial. If M is nonorientable, apply the above to the oriented double cover.

If M has odd dimension and is nonorientable, let $\overline{M}$ be the oriented double cover with covering metric. Then an isometry = deck transformation must reverse orientation since M is nonorientable. By the above, it has a fixed point. Thus, every deck transformation is trivial, so $\Pi_1(M) = 0$, contradicting non-orientability. $\square$

Theorem 14.9. *One useful lemma was the following: if c is not a null-homotopoic loop on compact M and $[c]$ denotes it's (free) homotoply class, then there is a geodesic γ with $[\gamma] = [c]$. One easy proof of this is to look in the universal covering space, and use the fact that M compact $\Rightarrow$ M complete$\Rightarrow$ universal covering space is complete with the covering metric, to create a geodesic joining the endpoints of the lift of c. We must then only worry that $\pi \circ \tilde{c}$ is smooth at $\pi \circ \tilde{c}(a) = \pi \circ \tilde{c}(0)$.*

15 Rauch's Comparison Theorem

We say earlier that Jacobi fields measure how fast geodesics spread apart, given the curvature near a point. We obtained explicit formula which are good for small t which indicated that the bigger the curvature, the smaller the Jacobi field. The Rauch Comparison theorem roughly says that this continues to hold for long t.

Theorem 15.1. *(Rauch Comparison Theorem)Let* M *and* M^* *be n-dimensional manifolds. Let* $\gamma : [0,a] \to M$ *and let* $\gamma^* : [0,a] \to M^*$ *be unit speed geodesics. Let* $J(t)$ *and* $J^*(t)$ *be Jacobi fields with* $J(0) = J^*(0) = 0$*, and with* $J'(0)$ *and* $J'^*(0)$ *unit vectors orthogonal to their respective curves. Suppose that* $K(x, \gamma') \leq K^*(x^*, \gamma'^*)$ *for any* x *and* x^* *in the right tangent spaces, for any* t*. Finally, suppose that* γ^* *has no conjugate points on* $[0,a]$*. Then* $|J(t)| \geq |J^*(t)|$ *for all t. Further more, if they're in size at any point after* $t = 0$*, then we have* $K(J, \gamma') = K^*(J^*, \gamma'^*)$ *for all t.*

The proof relies heavily on the following:

Theorem 15.2. *(Index Lemma) Let* γ *be a geodesic with no conjugate points on* $[0,a]$*. Let* J *be a Jacobi field along* γ *and orthogonal to it, and let* V *be any other piecewise smooth vector field along* γ *and orthogonal to it. Suppose* $J(0) = V(0) = 0$ *and that* $\exists t_0$ *such that* $J(t_0) = V(t_0)$*. Then* $I_{t_0}(J,J) \leq I_{t_0}(V,V)$*. Equality holds iff* $V = J$ *on* $[0,t_0]$*. Here,* $I_{t_0}(X,X) = \int_0^a < V', V' > - < R(\gamma', V)\gamma', V > dt$*.*

Some quick applications of the Rauch comparison theorem are..

Theorem 15.3. *If the sectional curvature* K *satisfies* $0 < l \leq K \leq H$*, then the distance* d *between two consecutive conjugate points along a geodesic satisfy* $\pi/\sqrt{H} \leq d \leq \pi/\sqrt{K}$*. This is proved by comparing with spheres.*

Theorem 15.4. *Let M and N are such so that all sectional curvatures in M are smaller than those in N, and let* $c : [0,a] \to M$ *be any curve who image lies in a normal ball. Let* $c^* = exp_{p^*} \circ i \circ (exp_p)^{-1} \circ c$*, with* i *any isometry. Then the lenght of* c *is* $\geq$ *the length of* c^**.*

Finally, we have 2 theorems about comparison, though I'm not exactly sure how it related to Rauch.

Theorem 15.5. *(Bishop Volume Comparison Theorem) Let* M *have Ricci curvature* $\geq K^*$ *everywhere. Let* M^* *be the complete simply connected manifold of the same dimension with constant sectional curvature* K^**. Let* $B_r(p)$ *and* $B_r^*(p^*)$ *denote normal balls of radius* r *in* M *and* M^* *respectively. Then* $Vol(B_r(p)) \leq Vol(B_r^*(p^*))$*.*

Theorem 15.6. *(Toponogov's Comparison Theorem) Let* M *be a complte manifolds with* $K \geq K^*$*, where* K^* *is a constant. Let* M^* *be the complete simply connected maniofld with constant curvature* K^**. Let* γ_1 *and* γ_2 *be geodesic segments beginning at a common point* p *and making an angle of* θ*, having lengths* l_1 *and* l_2*, and ending at* p_1 *and* p_2*. Do the same setup (with *) on* M^**. Assume the* γ_1 *is minimizing and if* $K^* > 0$*, assume also that* $l_2 \leq \pi/\sqrt{K^*}$*. Then,* $d(p_1, p_2) \leq d(p_1^*, p_2^*)$*.*

I know almost nothing about the Bishop thereom, but heres a quick applications of of Toponogrov's theorem.

Theorem 15.7. *(Cheeger-Perelman). If* M *has non-negative sectional curvature, the there exists a compact totally geodesic submanifold* S*, called the Soul of* M*, such that* M *is diffeomorphic to the normal bundle of* S*. If* M *is compact, the* $S = \{pt\}$*.*