Everything you wanted to know about G_2 and a lot you didn't.

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Abstract

I've compiled facts coming mostly handwritten notes by Wolfgang Ziller and partly from a paper by Herman Gluck, Frank Warner, and Wolfgang Ziller. These facts, hopefully, will shed some light on what the exceptional Lie group G_2 is, both as a group and topologically. Further, we'll use it to write 3 spheres as homogenous spaces in unique ways. Finally, we'll analyze the subgroups of G_2 up to conjugacy.

Contents

1 Introduction

We'll start by describing the Cayley numbers, then we'll go over lots of properties of them, and then we'll define G_2 and figure out some properties of it.

First, given $\mathbb{R}$ we can write $\mathbb{C}$ as

$$\mathbb{C} = \mathbb{R} + \mathbb{R}$$

with $(a, b)(c, d) = (ac - bd, ad + bc)$. Since the reals are commutative and associative, no more care is needed. Similarly, we can write

$$\mathbb{H} = \mathbb{C} + \mathbb{C}$$

where we think of (z_1, z_2) as $z_1 + z_2 j$. Then we have $(z_1, z_2)(w_1, w_2) = (z_1 w_1 - z_2 \overline{w_2}, z_1 w_2 + z_2 \overline{w_1})$.

Then, we similarly write

$$Ca = \mathbb{H} + \mathbb{H}$$

where we think of (q_1, q_2) as $q_1 + q_2 l$. Then multiplication is given by $(q_1, q_2)(s_1, s_2) = (q_1 s_1 - \overline{s_2} q_2, s_2 q_1 + q_2 \overline{s_1})$. To see that is correct, notice that when multiplying it out, the fundamental question will be to compute $(q_2 l)(s_2 l)$. Remember that things are NOT associative, so a lot of care must be taken. By just playing around with this, one can prove that $(q_2 l)(s_2 l) = -\overline{s_2} q_2$.

We now begin the definitions/easy lemmas.

Definition 1.1. The conjugate of the Cayley number (q_1, q_2), denoted $\overline{(q_1, q_2)}$ is defined as $\overline{(q_1, q_2)} = (\overline{q_1}, -q_2)$.

Lemma 1.1. $\overline{(a, b)(c, d)} = \overline{(c, d)}\;\overline{(a, b)}$

Proof.

$$\overline{(a, b)(c, d)} = \overline{(ac - \overline{d}b, da + b\overline{c})} = (\overline{ac} - \overline{\overline{d}b}, -da - b\overline{c}) = (\overline{ca} - \overline{b}d, -da - b\overline{c})$$

while

$$\overline{(c, d)(a, b)} = (\overline{c}, -d)(\overline{a}, -b) = (\overline{ca} - \overline{b}d, -b\overline{c} - d\overline{\overline{a}}) = (\overline{ca} - \overline{b}d, -b\overline{c} - da)$$

so they're equal. □

Lemma 1.2. $\overline{\overline{(a, b)}} = (a, b)$

Proof. $\overline{\overline{(a,b)}} = \overline{(\overline{a},-b)} = (\overline{\overline{a}}, --b) = (a,b)$ ☐

Definition 1.2. $Re(a,b) = (Re(a),0)$
$Im(a,b) = (Im(a),b)$

Lemma 1.3. $Re(a,b) = 1/2(a,b) + 1/2\overline{(a,b)}$

Proof. $1/2(a,b) + 1/2\overline{(a,b)} = 1/2(a,b) + 1/2(\overline{a},-b) = (1/2(a+\overline{a}),0) = (Re(a),0)$ ☐

Lemma 1.4. $(a,b)\overline{(a,b)} = (|a|^2 + |b|^2, 0) \in \mathbb{R}$.

Proof. First note that $\overline{(a,b)\overline{(a,b)}} = \overline{\overline{(a,b)}(a,b)} = (a,b)\overline{a,b}$, so that $(a,b)\overline{(a,b)}$ is real. Thus, we only compute the part in the first slot. There, we get $a\overline{a} - \overline{-b}b = |a|^2 + |b|^2$ as claimed. ☐

Definition 1.3. $<(a,b),(c,d)>= Re(a,b)\overline{(c,d)} =< a,c > |_{\mathbb{H}} + < b,d > |_{\mathbb{H}}$.

Theorem 1.5. $||(a,b)(c,d)||^2 = ||(a,b)||^2||(c,d)||^2$

Proof. We just compute it:

$||(a,b)(c,d)||^2 = ||(ac-\overline{d}b, da+b\overline{c})||^2 = (ac-\overline{d}b, da+bc)(\overline{ca}-\overline{b}d, -da-b\overline{c}) = ((ac-\overline{d}b)(\overline{ca}-\overline{b}d) - \overline{(-da-b\overline{c})}(da+b\overline{c}),0) = ((||a||^2+||b||^2)(||c||^2+||d||^2) + \overline{a}\overline{d}b\overline{c} + c\overline{b}da - ac\overline{b}d - \overline{d}b\overline{ca},0) = ((||a||^2+||b||^2)(||c||^2+||d||^2) + \overline{(c\overline{b}da)} + c\overline{b}da - (ac\overline{b}d + \overline{(ac\overline{b}d)}),0) = ((||a||^2+||b||^2)(||c||^2+||d||^2) + 2Re(c\overline{b}da) - 2Re(ac\overline{b}d),0)$ But $Re(xy) = Re(yx)$ so that the two Res cancel out. Thus we find that $||(a,b)(c,d)||^2 = ((||a||^2+||b||^2)(||c||^2+||d||^2),0)$.

Likewise, $||(a,b)||^2||(c,d)||^2 = ((||a||^2+||b||^2)(||c||^2+||d||^2),0)$ so the result is proved. ☐

2 Facts about the inner product

We prove a lot of simple, but important facts. First recall that if $x, y \in Ca$ then we know that $|x||y| = |xy|$ and $< x,y >= Re(x\overline{y}) = 1/2(x\overline{y} + y\overline{x})$.

Lemma 2.1. *Observe that if $|x| = 1$, then L_x and R_x are isometries since $||L_x(y)|| = ||xy|| = ||x||||y|| = ||y||$ and likewise for R_x. We can use this to show that $SO(8)$ is diffeomorphic to $SO(7) \times S^7$.*

Proof. Define $f : SO(7) \times S^7 \to SO(8)$ by

$$f(A,x) = L_x \circ \begin{bmatrix} 1 & 0 \\ 0 & A \end{bmatrix}$$

Define $g : SO(8) \to SO(7) \times S^7$ by $g(B) = (L_{B(1)}^{-1} \circ B, B(1))$. Then

$$f(g(B)) = f(L_{B(1)}^{-1} \circ B, B(1)) = L_{B(1)} \circ L_{B(1)}^{-1} \circ B = B$$

and

$$g(f(A,x)) = g(L_x \circ \begin{bmatrix} 1 & 0 \\ 0 & A \end{bmatrix}) = (L_x^{-1} L_x \begin{bmatrix} 1 & 0 \\ 0 & A \end{bmatrix}, x) = (\begin{bmatrix} 1 & 0 \\ 0 & A \end{bmatrix}, x)$$

Thus, $f \circ g$ and $g \circ f$ are the identity functions, so that we have a diffeomorphism. $\square$

Lemma 2.2.

$$< xw, yw >=< x, y > ||w||^2$$

and likewise

$$< wx, wy >=< x, y > ||w||^2$$

Proof. First note that if $x = y$, we have

$$< xw, xw >= ||xw||^2 = ||x||^2 ||w||^2 =< x, x > ||w||^2,$$

so it's true. Now, notice that

$$\begin{aligned} < x, x > ||w||^2 &+ 2 < x, y > ||w^2|| + < y, y > ||w||^2 \\ &= < x + y, x + y > ||w||^2 \\ &= < (x + y)w, (x + y)w > \\ &= < xw + yw, xw + yw > \\ &= < xw, xw > +2 < xw, yw > + < yw, yw > \\ &= < x, x > ||w||^2 + 2 < xw, yw > + < y, y > ||w||^2 \end{aligned}$$

Cancelling appropriate terms gives the result. $\square$

Lemma 2.3. *$< x, yw >=< x\overline{w}, y >$. In otherwords, $R_w^t = R_{\overline{w}}$. Likewise $L_w^t = L_{\overline{w}}$.*

Proof. First, notice that R_w^t is linear in w, and clearly if w is real, $R_w^t = R_w$. Thus, we can assume w is purely imaginary, i.e. $\overline{w} = -w$. Then notice that

$$< z, zw >=< 1, w > ||z||^2 = 0.$$

Thus, we have

$$0 =< x + y, (x + y)w >=< x + y, xw + yw >=< x, yw > + < y, xw >$$

so that

$$< x, yw >= - < y, xw >=< y, x(-w) >=< y, x\overline{w} >=< x\overline{w}, y > .$$

$\square$

Lemma 2.4. *If $x, y, \in ImCa$ and if $< x, y >= 0$, then $xy = -yx$. More generally if $< x, y >= 0$, then $x\overline{y} = -y\overline{x}$.*

Proof.

$$\begin{aligned} 0 &= < x, y > \\ &= Re(x\overline{y}) \\ &= 1/2(x\overline{y} + \overline{x\overline{y}}) \\ &= 1/2(x\overline{y} + y\overline{x}) \\ &= -1/2(xy + yx) \end{aligned}$$

so that $xy = -yx$. ☐

Lemma 2.5. *$(xy)\overline{y} = x||y||^2$ and $\overline{x}(xy) = ||x||^2 y$.*

Proof. $< (xy)\overline{y}, z >=< xy, zy >=< x, z > ||y||^2 =< x||y||^2, z >$. Since it's true for all z, we must have $(xy)\overline{y} = x||y||^2$. ☐

Lemma 2.6. *$x^{-1} = \overline{x}/||x||^2$ and if $xw = y$, then $w = x^{-1}y$.*

Proof. $x\overline{x}/||x||^2 = \overline{x}/||x||^2 x = 1$ and $x(\overline{x}/||x||^2 y) = 1/||x||^2 x(\overline{x}y) = y$. ☐

Lemma 2.7. *$x(\overline{y}w) + y(\overline{x}w) = 2 < x, y > w$ and likewise $(w\overline{y})x + (w\overline{x})y = 2 < x, y > w$.*

Proof. $2 < x, y > w - x(\overline{y}w) - y(\overline{x}w) = (x\overline{y} + y\overline{x})w - x(\overline{y}w) - y(\overline{x}w) = (x\overline{y})w + (y\overline{x})w - x(\overline{y}w) - y(\overline{x}w) = ((x+y)(\overline{x}+\overline{y}))w - (x+y)((\overline{x}+\overline{y})w) = 0$ by Lemma 5. ☐

Lemma 2.8. *If $A \subseteq Ca$ is any 4 dimensional subalgebra. Let $\epsilon \in A^{\perp}$ with $||\epsilon||^2 = 1$. Then $A\epsilon \perp A$ and hence $Ca = A + A\epsilon$. Further, $(a + b\epsilon)(c + d\epsilon) = (ac - \overline{d}b) + (da + b\overline{c})\epsilon$.*

Proof. $1 \in A$ (by definition) and hence $a \in A$ implies $\overline{a} \in A$. Thus, if $a, b \in A$, then $< a\epsilon, b >=< \epsilon, \overline{a}b >= 0$ so that $A\epsilon \perp A$.

To see the second part of the statement, we'll need to compute $(b\epsilon)(d\epsilon)$. In order to do this, notice first that since $\epsilon \perp \mathbb{R}$, we have $\overline{\epsilon} = -\epsilon$. Next notice that $0 = 2 < b, \epsilon >= b\overline{\epsilon} + \epsilon\overline{b}$. Thus, we have $b\epsilon = \epsilon\overline{b}$ and hence, we have that $(b\epsilon)(d\epsilon) = (\epsilon\overline{b})(d\epsilon) = 2 < d\epsilon, \overline{b} > -(\epsilon\overline{(d\epsilon)})b = -(\epsilon(\overline{\epsilon}\overline{d}))b = -\overline{d}b$.

Likewise, $(b\epsilon)c = -(b\overline{\epsilon})c = (b\overline{c})\epsilon$

☐

Lemma 2.9. *If A is a subalgebra and is generated by two elements x and y, then A is associative. Further, A is isomorphic to either $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$.*

Proof. If $Im(x) = Im(y) = 0$, then x and y clearly generate $\mathbb{R}$. In fact, since we assume $1 \in A$, we can assume wlog that $Re(x) = Re(y) = 0$ Further, by replacing x by $x/||x||^2$, and similarly for y, we can assume $||x|| = ||y|| = 1$. Then $\overline{x} = -x$ and so $x^2 = -x\overline{x} = -||x||^2 = -1$. Thus, clearly x generates an algebra isomorphic to $\mathbb{C}$. Now, either y is in this algebra or not. If y is in it, we, of course, get $\mathbb{C}$. If y is NOT in this algebra, let $y = y_1 + y_2$ where y_1 is the part in the algebra and y_2 is orthgonal to it. Then A is generated by x and y_2, and we can assume wlog that y_2 has length 1. Then, one can show this is isomorphic to $\mathbb{H}$ without too much effort. Thus, we see that A is isomorphic to an associative algebra, and hence, is associative. □

Finally, we come to the heart of the matter - this will help us figure out what G_2 is.

Lemma 2.10. *Let e_1, e_2, and e_3 be an o.n. triple in $ImCa$ with $e_3 \perp e_1e_2$. Then there is a unique $\mathbb{R}$ linear isometry $B : Ca \to Ca$ such that $B(xy) = B(x)B(y)$, $B(1) = 1$, $B(e_1) = (i, 0)$, $B(e_2) = (j, 0)$, and $B(e_3) = (0, 1)$.*

Proof. First, if B exists, it's an isometry since $B(1) = 1$, so it preserves the real part and if x is imaginary, we have $-||Bx||^2 = -Bx\overline{Bx} = BxBx = B(x^2) = B(-||x||^2) = -||x||^2$. Thus, B is an isometry.

To see that B exists and is unique, let A be the algebra generated by e_1 and e_2. Then, by Lemma 9, clearly $A \cong \mathbb{H}$. We can obviously start by defining B on A by $B(1) = 1$, $B(e_1) = (i, 0)$, $B(e_2) = (j, 0)$, and $B(e_1e_2) = (k, 0)$. This is clearly a well defined isomorphism from A onto $\mathbb{H}$. Since $e_3 \perp e_1e_2$, we can let $\epsilon = e_3$ and apply Lemma 2.8 to see that Ca is generated by e_1, e_2, and e_3. Thus, if we simply define $B(a + b\epsilon) = B(a) + B(b)B(\epsilon) = B(a) + B(b)(0, 1)$, then this defines B on all of Ca and satisfies the product formula since

$$B((a + b\epsilon)(c + d\epsilon)) = B((ac - \overline{d}b) + (da + b\overline{c})\epsilon)$$

$=$

$$B(ac-\overline{d}b)+B(da+b\overline{c})B(\epsilon) = B(a)B(c)-\overline{B(d)}B(b)+(B(d)B(a)+B(b)\overline{B(c)})B(\epsilon)$$

while

$$B(a + b\epsilon)B(c + d\epsilon) = (B(a) + B(b)B(\epsilon))(B(c) + B(d)B(\epsilon))$$

$=$

$$(B(a)B(c) - \overline{B(d)}B(b)) + (B(d)B(a) + B(b)\overline{B(c)})B(\epsilon)$$

so that we have $B(xy) = B(x)B(y)$. Uniqueness is clearly forced by linearity. □

3 G_2 stuff

Lemma 3.1. *The set of automorphisms of Ca is denoted by $G_2 \subseteq SO(8)$ and is a simple, centerless Lie Group of dimension 14 which is compact, connected, and simply connected.*

Proof. It's clear that G_2 is a closed subgroup of $SO(8)$ since it's defined by an equation, and is thus a compact Lie group (in fact, embedded in $SO(8)$ by the closed subgroup theorem). Further, since $B(1) = 1$ for any $B \in G_2$, $G_2 \subseteq SO(7)$ To see connectedness, simply connectedness, and the dimension count, consider the two fibrations

$$S^3 \to G_2 \to V_2(\mathbb{R}^7)$$

and

$$S^5 \to V_2(\mathbb{R}^7) \to S^6$$

where the two projection maps are $(e_1, e_2, e_3) \to (e_1, e_2)$ and $(e_1, e_2) \to e_1$. We use the second to gain information about $V_2(\mathbb{R}^7)$. In particular, since both S^5 and S^6 are 4-connected, it follows from the LES of homotopy groups that $V_2(\mathbb{R}^7)$ is 4-connected as well. Thus, the LES homotopy sequence for the G_2 fibration tells us that $\Pi_3(S^3) = \mathbb{Z} = \Pi_3(G_2)$ so that G_2 is simple. Further, since S^3 is 2-connected, it follows from this same LES that G_2 is 1-connected (and, in fact, is 2-connected since ALL Lie groups have $\Pi_2 = 0$).

To see it's centerless, let A be in the center. Note that if $B \in G_2$ and B has a fixed point x, then B must also fix Ax since $Ax = ABx = BAx$. Let B be the transformation which takes i to i, j to j and l to il. Then, since B fixes i and j, B must fix Ai and Aj. Thus, we see that Ai and $Aj \in \mathbb{H} \cup span jl, kl$. By letting B take l to jl or l to kl, we see that Ai and $Aj \in \mathbb{H}$. It follows that $A(\mathbb{H}) = \mathbb{H}$ and since A is an isometry, that $A(\mathbb{H}l) = \mathbb{H}l$.

Now, let B be the element of G_2 taking i to i, j to k and l to l. Then since B fixes i, we must have B fixing Ai, but this will force $Ai = \pm i$. Similarly, by choosing B to take i to k and fixing j, we find that $Aj = \pm j$. Thus, we find that $Ak = \pm k$ as well. Notice, we must have an odd number of +1s here. Including the fact that $A(1) = 1$, we must have either 2 or 4 +1s in the matrix for B in the part corresponding to $\mathbb{H}$.

Next, by letting B send i and j to j, i; j, k; or k, i, , while keeping l fixed, we see that $Al = \pm l$. It follows from this that Ail, Ajl, and Akl are all ± 1, though once we determine the sign for i, j, and l, the rest are determined.

Now, assuming $A(i) = -i$, let B send i to j, j to i, and fix l. Then $A(B(i)) = A(j) = \pm j$ and $B(A(i)) = B(-i) = -j$ and assuming $A(i) = i$, we find $A(i)$ and $A(j)$ ahve the same sign. Similarly, letting B swap i and k while fixing l shows that $A(i)$ and $A(k)$ have the same sign, so that $A(i)$, $A(j)$, and $A(k)$ have the same sign (which must therefore be positive). Now, let B be defined by swapping i and l while fixing j. Then we find that $A(i)$ and $A(l)$ have the same (positive) sign. Thus A is the identity.

$\square$

We'll show some interesting properties of SO(8) as it related to Ca now.

Theorem 3.2. *SO(8) is generated by* L_x *and* R_x*, with* $|x| = 1$ *and* $x \in Ca$*.*

Proof. I first claim that if $A \in SO(8)$ and $A(1) = 1$ (so $A \in SO(7)$), then A is a product of transformations of the form $R_{\overline{x}} \circ L_x$. To see this, notice that $R_{\overline{x}} \circ L_x(x) = xx\overline{x} = x$ and if $< x, y >= 0$ and $y \in ImCa$, then $R_{\overline{x}}L_x y = xy\overline{x} = -xyx = x^2y = (-x^2)(-y) = (x\overline{x})(-y) = -y$. Thus, we see that $R_{\overline{x}} \circ L_x$ is a reflection through x. Since we can obtain any reflection in $SO(7)$ like this, and since $SO(7)$ is generated by reflections, we can make A.

Next, I claim that if $u, v \in S^7 \subseteq Ca$ that we can take u to v via products of elements of teh form L_x. To see this, first note that if $< u, v >= 0$, then we have $0 =< u, v >=< 1, vu^{-1} >$ so $x = vu^{-1} \in ImCa$. Then $L_x(u) = (vu^{-1})u = v$.

If $< u, v > \neq 0$, then choose z orthogonal to both u and v and consider $L_{vz^{-1}} \circ L_{zu^{-1}}$. Then $L_{vz^{-1}} \circ L_{zu^{-1}}(u) = (vz^{-1})((zu^{-1})u) = (vz^{-1})(z) = v$ as desired.

Thus, $SO(8)$ is clearly generated by products of L_x and R_x as claimed. $\square$

Note 3.1. $\{L_x | x \in Ca\}$ is NOT a group, but it it does generate a subgroup of $SO(8)$. Similarly, $\{R_{\overline{x}} \circ L_x | |x| = 1\}$ is not a group, but generates $SO(7)$.

We now move onto the Moufang identities, which will let us understand Triality of $SO(8)$.

Lemma 3.3. $(xyx)z = x(y(xz))$*,* $z(xyx) = ((zx)y)x$ *and* $(xy)(zx) = x(yz)x$*.*

Proof. We may assume z is orthogonal to the algebra generated by x and y and that x and y are orthogonal. Then $xyx = (x\overline{\overline{y}})x = -(x\overline{x})\overline{y} = -|x|^2\overline{y}$. Thus, $(xyx)z = (-|x|^2\overline{y})z = -|x|^2(\overline{y}z)$.

Similarly, $x(y(xz)) = x(\overline{\overline{y}}(xz)) = -\overline{y}(\overline{x}(xz)) = -\overline{y}|x|^2z = -|x|^2\overline{y}z$, where the second equality follows from Lemma 2.7. Thus, they are equal as claimed. The other two follow similarly. $\square$

We now move on the the cool stuff - triality.

Theorem 3.4. *Consider $A, B, C \in SO(8)$ satifsying $A(x)B(y) = C(xy)$ for all $x, y \in Ca$. Then given exactly one of A, B, and C, then the other two exist and are unique up to sign. In otherwords, if given A, (B, C) exists and the only ambiguity is of the form $\pm(B, C)$. Similarly, if given C, (A, C) exists and is unique up to $\pm(A, C)$.*

Proof. First, notice that any of the following three triples of automorphisms work: $(R_z, L_zR_{\overline{z}}, R_z)$, $(L_zR_{\overline{z}}, L_z, L_z)$, and $(L_z, R_z, -L_zR_{\overline{z}})$. All three follow from the Moufang identities. For example, $(R_z(x))(L_zR_{\overline{z}}(y)) = (xz)(zy\overline{z}) = -(xz)(zyz) = -(((xz)z)y)z = (((zx)(-z))y)z = (((xz)\overline{z})y)z = (xy)z = R_z(xy)$.

Now, let $G = \{C|(A, B) \text{ exist}\}$. I claim that G is a group since if $C \to (A, B)$ and $C' \to (A', B')$, then $C \circ C' \to (A \circ A', B \circ B')$ since $C(C'(xy)) = C(A'(x)B'(y)) = A(A'(x))B(B'(y))$.

But then, G contains R_z and L_z, so that $G = SO(8)$ (since L_z and R_z generate $SO(8)$).

Now, given C, what choice of A and B is there? We may as well assume $C = Id$. Then we have $A(x)B(y) = xy$ for all x and y. Plugging in $y = 1$ yields $A(x)B(1) = x$ so that $A(x) = xa = xB(1)^{-1}$ for all x. Similarly, we find that $B(y) = a^{-1}y$.

Then, $xy = A(x)B(y) = (xa)(a^{-1}y) =$(by Lemma 2.7)$-a((\overline{a}x)y) = xy$ so that $(\overline{a}x)y = \overline{a}(xy)$. Since this is true for all x and y, it follows that a is actually real.

Now, since $A(1) = a \in R$, it follows that $a = \pm 1$, so that $A = \pm Id$ and likewise for B. Since it's clear that $(A, B) = -(Id, Id)$ works, we're done.

Now, if $A = Id$, what choice in B and C do we have? (This argument works similarly if given $B = Id$.) $A = Id$ tells us that $xB(y) = C(xy)$ for all x and y. Plugging in $x = 1$ gives us $B(y) = C(y)$. If $y = 1$, we get $xB(1) = xa = C(x) = B(x)$. Then $x(B(y)) = C(xy)$ so $x(ya) = (xy)a$, from which it follows that a is real, so that $a = \pm 1$, so that $B = \pm Id$ and likewise C.

□

Note 3.2. Notice that $A \in G_2$ precisely when $A(x)A(y) = A(xy)$, i.e., $A = B = C$.

Lemma 3.5. *We can use this to give nice descriptions of Spin(8) and Spin(7)*

Proof. Let $G = \{(A, B)|\exists C\}$. Then G is connected since it's generated by the 3 kinds of "'basic"' triples of automorphisms, seen in the previous proof (and each one of those is clearly connected). Further, there is a natural map $G \to SO(8)$ given by $(A, B) \to C$. We've already established this is a surjective homomorphism. Further, we've also shown its kernel is $\pm Id$. It follows that $G = Spin(8)$.

Similarly, let $H_1 = \{(A, C) | \exists B \text{ and } A = C\}$, $H_2 = \{(B, C) | \exists A \text{ and } B = C\}$. Notice real quick that, in the case of H_1, we'll have that $B(1) = 1$ since $A(x)B(1) = A(x)$. Thus, we get a map from $H_1 \to SO(7)$ and like above, it will be a two fold cover. Similarly for H_2. Also, H_1 is clearly generated by (R_z, R_z) and H_2 is generated by (L_z, L_z) and hence both are connected. Thus, we see that L_z generated a $Spin(7) \subseteq SO(8)$ and the R_z generate another one. Further, it's clear that G_2 is precisely the intersection of H_1 and H_2. □

Another interesting fact is the following:

Lemma 3.6. *H_1 and H_2 are NOT conjugate, but there is an automorphism of $SO(8)$ taking H_1 onto H_2.*

Proof. Let $K : Ca \to Ca$ be the map which sends $x \to \overline{x}$. Notice that $KK(x) = \overline{\overline{x}} = x$ so that $K^2 = Id$. Now, define $f : SO(8) \to SO(8)$ by $f(A) = KAK$. Then $f(L_z)(x) = KL_zK(x) = K(z\overline{x}) = x\overline{z} = R_{\overline{z}}(x)$ so that f takes L_z to $R_{\overline{z}}$. It follows that f takes generates to generators so that H_1 goes to H_2.

To see these are not conjugate, just notice that in terms of a matrix stuff, f multiplies the first row and first column by -1. Thus, it takes $diag(R(\theta), 1, 1, 1, 1, 1, 1)$ to $diag(R(-\theta), 1, 1, 1, 1, 1, 1)$, so this clearly isn't conjugation by an orientation preserving element.

□

4 Spheres written using G_2

We now come to part of the point of this paper.

Theorem 4.1. $G_2/SU(3) = S^6$.

Proof. We know $G_2 \subseteq SO(7)$ so it naturally acts on S^6. Further, I claim that it acts transitively. To see this, I claim we can take i anywhere. To see this, let e_1 be arbitrary, and choose e_2 perpendicular to e_1 and e_3 perpendicular to e_1, e_2,and e_1e_2. Then, we know there is a unique $A \in G_2$ taking $i \to e_1$, $j \to e_2$, and $l \to e_3$. In particular, we have $A(i) = e_1$ so that G_2 acts transitively.

Now, we compute the isotropy at i. So, suppose $A \in G_2$ and $A(i) = i$. Notice, then $A \subseteq SO(6)$. Further, I claim $A \circ L_i = L_i \circ A$. So see this, just notice that $A(ix) = A(i)A(x) = iA(x) = L_i(A(x))$. Further, $L_i : \mathbb{R}^6 \to \mathbb{R}^6$ is a complex structure (Here, $\mathbb{R}^6 = span\{1, i\}^\perp$). Thus, we have that $A \in SO(6) \cap GL(3, \mathbb{C}) = U(3)$. Next, notice that the isotropy group is 1-connected since S^6 is 2-connected and G_2 is 2-connected (use the LES in homotopy groups). Thus, the isotropy group is a 1-connected 8 dimensional subgroup of $U(3)$, that is, the isotropy group is a compact simply connected (hence product of simple lie groups) lie group of dimension 8. But from the classification of these, it's clear the only choice is $SU(3)$. □

Theorem 4.2. $Spin(7)/G_2 = S^7$

Proof. We write $H_2 = Spin(7) = \{(B, C) | \exists A \text{ and } B = C\}$, so that $A(1) = 1$. $Spin(7)$ acts transitively on S^7 since we've seen that this $Spin(7)$ is generated by (L_z, L_z), and we've already seen that the L_z act transitively on unit vectors.

We now compute the stabilizer of $1 \in S^7$. Notice that $B(1) = 1 = C(1)$ by definition of stabilizer. Thus we have $A(x)B(y) = C(xy)$ and plugging in $y = 1$ gives $A(x)B(1) = B(x) = A(x)$ so that $A = B = C$, i.e., $B \in G_2$. □

5 Spin(9)/Spin(7)

In this section, we show why $Spin(9)/Spin(7) = S^{15}$ for a particular embedding of $Spin(7)$ into $Spin(9)$. In order to do this, we'll have to take quite the detour and study the fibration $S^7 \to S^{15} \to S^8$. To this end, let $\mathbb{R}^{16} = Ca + Ca$. For each $m \in Ca$, let $L_m = \{(u, mu) | u \in Ca\}$. Let $L_\infty = \{(0, u) | u \in Ca\}$.

Note 5.1. Each L_m and L_∞ are real 8-dimensional subspaces, but they are NOT Cayley subspaces due to the lack of associativity fo the Cayley numbers.

Lemma 5.1. *The Cayley lines fill* $\mathbb{R}^{16}$ *and any two only intersect at the origin.*

Proof. Let $p = (u, v) \in Ca + Ca$. If $u = 0$, the clearly $p \in L_\infty$. Now, assume $u \neq 0$ and let $m = vu^{-1}$. Then $mu = (vu^{-1})u = v$ so that $p \in L_m$.

Now, it's clear that $L_m \cap L_\infty = 0$ since if $(u, v) \in L_m \cap L_\infty$, then $u = 0$ and $v = mu = 0$, so that $(u, v) = (0, 0)$.

If $(u, v) \in L_m \cap L_n$ with $m \neq n$, then $mu = v = nu$. Thus, $(m - n)u = 0$ so that $u = 0$. Then, since $v = mu = 0$, we have $(u, v) = (0, 0)$. □

Now, it is clear that the set of Cayley lines forms an S^8. This gives us a fibration $S^7 \to S^{15} \to S^8$ where the S^{15} is the unit sphere in $\mathbb{R}^{16} = Ca + Ca$, the S^7 is the unit sphere in each Cayley line.

It is a fact that this is in fact a Riemannian submersion if we let the S^7 and S^{15} be unit radius and the S^8 have radius 1/2.

Let $G = \{f : S^{15} \to S^{15} | f \text{ takes each fiber to each fiber set wise}\}$, that is G is the automorphism group of this fibration. Note that each $f \in G$ gives us an induced map on S^8 since each f preserves fibers.

We'll start studying the way G acts on S^{15} to learn a lot about it. Then we'll show that G is actually isomorphic to $Spin(9)$. This then gives an action of $Spin(9)$ on S^{15} and this is exactly the one we want.

We'll establish all this in a series of lemmas.

Lemma 5.2. *If $f \in G$ and $f(L_m) = L_m \forall m \in Ca \cup \infty$, then $f = \pm Id$*

Proof. Given such an f, extend it linearly to $f : \mathbb{R}^{16} \to \mathbb{R}^{16}$. Now, by definition, f fixes L_0 and L_∞, so by linearity, $f(u, v) = (Au, Bv)$ for some $A, B \in O(8)$. Since f fixes $L_1 = \{(u, u)\}$, we have that $A = B$, so that $f(u, v) = (Au, Av)$. Now, since f fixes L_m in general, we have $f(u, mu) = (Au, A(mu)) = (Au, m(Au))$.

Thus, we have that $A(mu) = m(Au)$ for all $m, u \in Ca$. Plugging in $u = 1$ gives that $A(m) = m(A(1)) = ma$. Plugging this back in gives that $(mu)a = m(ua)$ for all $m, u \in Ca$. But this implies that a is real. Then, since $A \in O(8)$, we have that $A(1) = a = \pm 1$. Then we find that $A(m) = m * \pm 1$ so that $A = \pm Id$. $\square$

Lemma 5.3. *If A is any orientation preserving isometry of S^8 which fixes L_0, then there is an $f \in G$ such that f induces the map A*

Proof. We can treat A as if it lies in $SO(8)$ since it fixes L_0. Then, the crucial point is that such a map f exists iff we can find $B, C \in O(8)$ such that $A(m)B(u) = C(mu)$ for all $m, u \in Ca$.

To see these are equivalent, first assume such an f exists. Then, since A is an isometry, A must also fix L_∞. This means that $f(L_0) = L_0$ and $f(L_\infty) = L_\infty$ so that we can write $f(u, v) = (Bu, Cv)$. Since $f \in G$, we must have that $f(L_m) = L_{A(m)}$ for each m. But then this says that $f(u, mu) = (Bu, C(mu)) = (Bu, A(m)(Bu))$., so that $C(mu) = A(m)B(u)$ as claimed.

Conversely, given A and assuming B and C exist, if we define $f(u, v) = (Bu, Cv)$, then this gives that $f(u, mu) = (Bu, C(mu)) = (Bu, A(m)B(u)) \in L_{A(m)}$ so that f preserves Cayley lines and clearly induces A.

But then the triality principle exactly tells us that given one of A, B, and C, we can always find the other two in order to make it work. Thus, given A, we can find B and C and the construct f as above. $\square$

Lemma 5.4. *G acts transitively on S^8*

Proof. Identify L_0 as the north pole and L_∞ as the southpole. Then clearly $\{L_m||m| = 1\}$ is the equator. Then $\{L_m|m\text{real }\}$ interesects the equator to two points, so it's clearly enough to show we can move L_0 to L_m for m real and abitrary.

To this end, let $A_\phi : Ca + Ca \to Ca + Ca$ be given by $A_\phi(u, v) = (\cos\phi u - \sin\phi v, \sin\phi u + \cos\phi v)$. I claim that $A_\phi \in G$. To see this, we need to show that $A_\phi(L_m) = L_{m'}$. So, given $(u, mu) \in L_m$, let

$$u' = (\cos\phi - \sin\phi m)u$$

and

$$m' = (\sin\phi + \cos\phi m)(\cos\phi - \sin\phi m)^{-1}$$

(notice the definition of m' depends only on m).

Then

$$m'u' = (\sin\phi + \cos\phi m)u$$

since we can reassociate here.

Thus, we have $A_\phi(u, mu) = (\cos\phi u - \sin\phi mu, \sin\phi u + \cos\phi mu) = (u', m'u') \in L_{m'}$, so that A_ϕ is in G.

But now notice $A_\phi(L_0) = L_{m'}$ where $m' = (\sin\phi + \cos\phi 0)(\cos\phi - \sin\phi 0)^{-1} = \tan\phi$, i.e., $A_\phi(L_0) = L_{\tan\phi}$. This clearly establishes the lemma. □

Note 5.2. At this point, we've shown that G induces an entire $SO(9)$ action on S^8.

Lemma 5.5. *No $f \in G$ induces an orientation reversing isometry of S^8.*

Proof. Let $f \in G$ and assume f induces an orientation reversing isometry of S^8. By composing with appropriate elements of $SO(8)$ (and therefore in G), we may assume $f(L_0) = L_0, f(L_\infty) = L_\infty$, and such that the induced map on S^8, A, is given by $A(u) = \overline{u}$. Then, we've already seen that $f(u, v)$ must be given as $f(u, v) = (Bu, Cv)$ with $(Au)(Bv) = C(uv)$.

Now, since f preserves Caylet lines, consider the equation $C(mu) = A(m)B(u) = \overline{m}(Bu)$. Plugging in $m = 1$ gives $C(u) = 1B(u)$. Since u is arbitrary, we conclude $C = B$.

Thus, we have that $C(mu) = \overline{m}C(u)$. Plugging in $u = 1$ gives $C(m) = \overline{m}C(1) = \overline{m}a$.

Thus, we have that $C(mu) = A(m)C(u)$ gives us $(\overline{mu})a = \overline{m}(\overline{u}a)$, i.e., that $(\overline{u}\overline{m})a = \overline{m}(\overline{u}a)$ for all m and u. By replacing $\overline{m}$ with m and $\overline{u}$, with u, we obtain

$$(um)a = m(ua)\text{for all } m, u \in Ca$$

I claim this is impossible. To see this, for consider the case that a is real. Then we can reassociate and see that $um = mu$ for all m and u in Ca, which is, of course, absurd.

If a is NOT real, then choose $m \in Ca$ such that $m \perp a$ with $m \notin \mathbb{R}$. Then m and a together generate an algebra isomorphic to $\mathbb{H}$. Thus, by restricting the values of u, we may assume wlog that $(um)a = m(ua)$ for all $u, m \in \mathbb{H}$. But then we CAN reassociate and cancel the a to obtain $um = mu$, which is, of course, absurd in $\mathbb{H}$. Thus, no such f can exist. □

Lemma 5.6. *$G = Spin(9)$*

Proof. Define $\phi : G \to SO(9)$ by $f \in G$ goes to its induced action A on S^8. By Lemma 5.5, $\phi(G) \subseteq SO(9)$ (that is, it doesn't land in $O(9)$). By Lemmas 5.4 and 5.3, ϕ is onto $SO(9)$. By Lemma 5.2, the kernel of ϕ is $\pm Id$. It follows that G is a double cover of $SO(9)$, so we must only show it's nontrivial.

To see this last fact, notice that Lemma 3.5 tells us that the $f \in G$ which fix L_0 is a connected set (in fact, isomorphic to $Spin(8)$). Then, since G is genrated by this $Spin(8)$ as well as the A_ϕ, which also form a connected set, it follows that G is connected, and hence, we have found the nontrivial double cover of $SO(9)$. Thus, $G = Spin(9)$ as claimed. □

Now that we know $G = Spin(9)$, we see how $Spin(9)$ acts on S^{15}.

Lemma 5.7. *Spin(9) acts transitively on S^{15}*

Proof. By Lemma 5.4, $Spin(9)$ can move any S^7 fiber to any other, so we need only show that $Spin(9)$ works transitively on one fiber, say L_0. Let $(p, 0), (q, 0) \in L_0 \subseteq Ca + 0$ and let $B \in SO(8)$ with $B(p) = q$. By triality, find A and C such that $A(u)B(v) = C(uv)$ and define $f : \mathbb{R}^{16} \to \mathbb{R}^{16}$ by $f(u, v) = (Bu, Cv)$. I clain that $f \in Spin(9)$ and that $f(p, 0) = (q, 0)$. To see that $f \in Spin(9)$, just notice that by the usual triality trick, $f(L_m) = L_{A(m)}$, so that $f \in G$ and $f(p, 0) = (Bp, C0) = (q, 0)$ as desired. □

Note 5.3. What we actually showed is that any map from one fiber to another can be extended to an $f \in G$.

Thus, we now know that $S^{15} = Spin(9)/H$ where H is the stabilizer of a point, say $(1, 0) \in Ca + Ca$. Further, notice by the LES of homotopy groups, since S^{15} and $Spin(9)$ are both simply connected and connected, H must be connected. Our goal now is to show that $H = Spin(7)$

Lemma 5.8. *$H = Spin(7)$*

Proof. We'll think of H as the stabilizer of $(1,0)$. If $f \in H$, then $f(L_0) = L_0$. Since f is an isometry, this also implies that $f(L_\infty) = L_\infty$, so that once again we can write $f(u,v) = (Bu, Cv)$. Since $B(1) = 1$, we have that $B(ImCa) = ImCa$.

Thus, we can define $\phi : H \to SO(7)$ by $\phi(f)$ is the induced action of B on $S^6 \subseteq ImCa$.

This is onto by the above remark. That is, given a map in $B \in SO(7)$, we can extend it as in the previous lemma to a map $f \in Spin(9)$. It's kernel is a two element set, and this can be seen as follows: Let $f \in ker\phi$ so that f fixes L_0 pointwise. But we've already seen that we can extend B to a map f of S^{15} only by using triality. Further, the triality principle asserts that there are precisely two such extensions. Those two obviously make up the kernel of ϕ.

Thus, we see that H is a double cover of $SO(7)$. But we've already seen that H is connected, so that H is $Spin(7)$ as claimed. □

6 Subgroups of G_2

In this section, we do our best to classify all the subgroups of G_2 up to conjugacy. For the simply subgroups, we'll do our best to compute the index, though to be honest, the representation theory neccesary is a bit difficult for me at this time.

To begin classifying the subgroups, we'll start with the rank 2 subgroups, since by the $Borel - Siebenthal$ theorem, these aren't too bad (since G_2 has rank two, which I haven't shown).

I didn't work out the details myself, but apparently the result is that there are only two choices for rank 2 subgroups of G_2, an $SU(3)$ that we've already seen and an $SO(4)$ which I'll describe now.

Lemma 6.1. *For $(q_1, q_2) \in Sp(1) \times Sp(1)$, define $A : Sp(1) \times Sp(1) \to G_2$ by $A(u,v)(q_1,q_2) = (q_1 u\overline{q_1}, q_2 v\overline{q_1})$. Here, we think of $(u,v) \in \mathbb{H} + \mathbb{H} = Ca$. I claim that $A \in G_2$.*

Proof. Compute: $A((u,v)(u',v')) = A(uu' - \overline{v'}v, v'u + v\overline{u'}) = (q_1(uu' - \overline{v'}v)\overline{q_1}, q_2(v'u+v\overline{u'})\overline{q_1})$, while $A(u,v)A(u',v') = (q_1u\overline{q_1}, q_2v\overline{q_1})(q_1u'\overline{q_1}, q_2v'\overline{q_1}) = (q_1uu'\overline{q_1} - q_1\overline{v'}\overline{q_2}q_2v\overline{q_1}, q_2v'\overline{q_1}q_1a\overline{q_1} + q_2v\overline{q_1}q_1\overline{u'}\overline{q_1})$, so that they agree. □

Note 6.1. Notice this gives an action of $SO(4)$ on $Im\mathbb{H}+\mathbb{H}$ because the kernel of this map from $Sp(1) \times Sp(1)$ into G_2 is $\pm Id$, and $Sp(1) \times Sp(1)/ \pm Id = S^3 \times S^3/ \pm Id = SO(4)$.

Now, what are the subgroups of $SU(3)$? The following are obvious: $SU(2) \subseteq U(2) \subseteq SU(3)$. Further, there is an $SO(3) \subseteq SU(3)$ obtained by taking a real matrix and just copying the entries.

Theorem 6.2. *Up to conjugacy, these are all the subgroups of* $SU(3)$.

What are the subgroups of $SO(4)$? It's obvious that there is an $SU(2) \subseteq U(2) \subseteq SO(4)$ by the usual identification of C^2 with $\mathbb{R}^4$. Further, there is clearly a $SO(3) \subseteq SO(4)$. I claim there is another $SU(2) \subseteq U(2) \subseteq SO(4)$.

Lemma 6.3. *There are two different* $U(2) \subseteq SO(4)$ *and the* $SU(2)$ *in each one are also distinct.*

Proof. On the $U(2)$ level, the nonstandard embedding is given by restricting the above action of $Sp(1) \times Sp(1)$ to elements of the form (q_1, z) with $z \in \mathbb{C}$.

To see the SU(2)s (and hence the U(2)s are distinct) is most easily seen by restricting the above map of $SO(4) \subseteq G_2$ to both $\{(1, q_2)\}$ and $\{(q_1, 1)\}$. In terms of actions, the first is $(u, v) \to (u, q_2 v)$ while the second is $(u, v) \to (q_1 u \overline{q_1}, v\overline{q_2})$. In terms of representations, the first is $3Id + \mathbb{R}^4$ while the second is $\mathbb{R}^3 + \mathbb{R}^4$.

We'll show these are nonconjugate by computing the index later. □

We now have another theorem that I don't know how to prove.

Theorem 6.4. *There are precisely 2* $U(2)$*s, 2* $SU(2)$*s and 1* $SO(3)$ *inside of* $SO(4)$ *up to conjugacy.*

Now, it turns out we have

Lemma 6.5. *The* $SO(3)$ *inside of* $SU(3)$ *and the* $SO(3)$ *inside of* $SO(4)$ *are the same.*

Proof. To see this, we argue as follows. The $SO(3)$ in $SO(4)$ looks like things of the form (q_1, q_1) acting on $Im\mathbb{H} + \mathbb{H}$. But then this fixes $(0, 1) = l$ and also commutes with L_l. But the $SU(3) \subseteq G_2$ was the stabilizer of a point (say, l), so that this $SO(3)$ is contained in $SU(3)$ as claimed.

□

We're almost done talking about $SO(3)s$ in G_2

Theorem 6.6. *There is precisely one other* $SO(3)$ *in* G_2 *which isn't contained in either* $SO(4)$ *or* $SU(3)$.

Proof. We know that $G_2 \subseteq SO(7)$, so we can ask what possible $SO(3) \subseteq SO(7)$ there are, and then see which lie in G_2. Now, the irreps of $SO(3)$ are classified - there is precisely one in each odd dimension. Thus, the possible 7 dimensional reps of $SO(3)$ are $\mathbb{R}^3 + 4Id$, $\mathbb{R}^5 + 2Id$, $\mathbb{R}^7$ and $\mathbb{R}^3 + \mathbb{R}^3 + Id$. Now, here's the key point: G_2 cannot fix just two vectors for then it will also fix their product. Further, G_2 cannot fix just 4 vectors, for then it will fix everything. Thus, our only choices are $\mathbb{R}^7$ and $\mathbb{R}^3 + \mathbb{R}^3 + Id$. This shows that there is AT MOST one other $SO(3)$ in G_2. Now, we've already seen the $\mathbb{R}^3 + \mathbb{R}^3 + Id$ is in $SO(4) \cap SU(3)$, so what about the other $SO(3)$? Turns out, it IS in G_2 though I don't know how to show it. ☐

Almost done:

Lemma 6.7. *The* $U(2) \subseteq SU(3)$ *is conjugate to one of the* $U(2)$ *in* $SO(4)$.

Proof. The $U(2)$ which comes from elements of the form (z, q_2) with $z \in C$ is in $SU(3)$. We can see this as follows: these fix $i = (i, 0)$ since $(zi\overline{z}, q_2 0\overline{z}) = (i, 0)$. But again, the $SU(3)$ was precisely the point stabilizer, so that we see this $U(2)$ is also contained in $SU(3)$. ☐

We'll now compute the index of all of these subgroups in G_2 (except the $SO(4)$). Recall that the index of a map $f : H \to G$, with H and G compact, simple Lie groups is defined by the degree of the induced map on Π_3, since Π_3 of a compact simple lie group is a copy of $\mathbb{Z}$. Also, from the LES of homotopy groups, $H \to G$ has degree k iff $\pi_3(G/H) = \frac{\mathbb{Z}}{k\mathbb{Z}}$.

To easy our life, notice the following 2 facts: degrees of maps multiply and the degree of G_2 in $SO(7)$ is 1. The first is a standard homotopy theory fact. To see the second, recall that $S^7 = Spin(7)/G_2$ and this double covers $SO(7)/G_2$. Since $pi_3(S^7) = 0$, we find that $k = 1$. Thus, computing the degree of a subgroup of G_2 is the same as computing its index in $SO(7)$.

First, I'll just state the following:

Theorem 6.8. *The* $SO(3) \subseteq G_2$ *which is NOT a subgroup of* $SO(4)$ *has index 28*

Now I'll do my best to compute the rest. We'll need a lemma.

Lemma 6.9. *If* H *has two representations* V_1 *and* V_2 *of index* k_1 *and* k_2, *then the representation* $V_1 + V_2$ *has index* $k_1 + k_2$.

Lemma 6.10. *The representation* $\mathbb{R}^3 + 4Id$ *of* $SO(3)$ *has index 2*

Proof. This representation is the usual $SO(3)$ in $SO(7)$. Then $SO(7)/SO(3)$ is the Stieffel manifold $V_4(\mathbb{R}^7)$ of 4 frames in $\mathbb{R}^7$. From Hatcher, we find that $\pi_3(V) = \frac{\mathbb{Z}}{2\mathbb{Z}}$, as claimed. ☐

Lemma 6.11. *The (standard) representation* $\mathbb{R}^4 + 3Id$ *of* $SU(2)$ *has index 1.*

Proof. $SU(2) \subseteq SU(3)$ is the standard embedding. The quotient is a 5-sphere, and thus the index is 1. The $SU(3) \subseteq G_2$ gives an S^6 as a quotient and hence has index 1. Thus, the $SU(2)$ has index 1. □

Note 6.2. We saw that our two $SU(2)$ representations were of them form $\mathbb{R}^3 + \mathbb{R}^4$ and $3Id + \mathbb{R}^4$. We can now compute in the index of these to be 3 and 1 respectively.

Thus, the only index left to compute is the $SO(3)$ in $SO(4) \cap SU(3)$. But this $SO(3)$ representation is $\mathbb{R}^3 + \mathbb{R}^3 + Id$ which has index 4.

7 References

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