Abstract

It is well known that if a complete manifold M has non-positive sectional curvature K, then M is covered by $\mathbb{R}^n$. One may attempt to improve this by asking whether there is a positive constant ϵ which only depends on the dimension n such that if M is a Riemannian n-manifold with $Kd^2 < \epsilon$, then M is covered by $\mathbb{R}^n$ (here d, denoting the diameter, is added to form a quantity independent of scaling). It turns out, the answer is no and a counterexample is given by S^3. That is, for every $\epsilon > 0$, there is a metric on S^3 such that $Kd^2 < \epsilon$. By normalizing the curvature to have a maximum of 1, this says that we can give S^3 a metric with curvature bounded above by 1, with diameter as small as we wish. This talk with follow a paper by Buser and Gromoll, which itself arose from a single sentence in a paper by Gromov as well as a letter by Gromov.

1 Introduction

In 1978, Gromov[3] proved that $\exists\ \epsilon = \epsilon(n)$ such that if M^n is any Riemannian manifold satisfying $|K|d^2 < \epsilon$, then M is covered by a nilpotent Lie Group. Since every nilpotent Lie group is diffeomorphic to $\mathbb{R}^n$, this, in particular, implies that M is covered by $\mathbb{R}^n$. One may wonder whether this is some $\epsilon' = \epsilon'(n)$ such that under the weakened hypothesis $Kd^2 < \epsilon'$, we may still conclude M is covered by $\mathbb{R}^n$. Of course, Hadamard's theorem also helps us to expect this. However, in the same paper (and in a single sentence), Gromov mentions that for any $\epsilon > 0$, there is a metric on S^3 satisfying $Kd^2 < \epsilon$.

Since then, Buser and Gromoll [2] have explicitely shown how to do this (apparentally following a letter of Gromov's). Further, Bavard [1] has shown how to achieve $Kd^2 < \epsilon$ on any closed orientable 3-manifold. We will show how Buser and Gromoll were able to do this.

A rough outline is given as follows:

First, we start with a large, round S^3 and a small $S^2 \times S^1$, where the S^2 is given a particular, non-round metric. We then find a closed geodesic in S^3 and then a comparable closed geodesics $S^2 \times S^1$. Our goal is then to drill out small tubular neighborhoods of each geodesic, and identify the remaining along their boundary to make a new manifold, which will be diffeomorphic to S^3. This will occur in 2 steps: 1. Before cutting out the geodesics, we use the Plumming lemma to deform each of the old metrics to be flat in a neighborhood of the geodesics, 2. Again, before cutting, we deform them further to put them in a nice form so that we glue smoothly and metrically.

This surgery will force some points near the geodesic in S^3 to all be close to each other, even if they were originally far away. Finally, we repeat the surgery enough (but still a finite number of) times on S^3, to make sure all points are close to some surgered geodesic.

It will turn out that step 1 may add some positive curvature, but we'll be able to estimate an upper bound for the curvature in terms of dimensional constants and an upper bound for the curvature of a small tubular neighborhood of the geodesic. Step 3 will in general add significant amounts of negative curvature (so that Gromov's theorem won't apply!).

The rest of the paper is organized as follows: Section 2 will offer an explanation as to why this is so nonintuitive - every metric on S^2 has the property that $Kd^2 \geq \pi^2$. Further, I'll attempt to carry out this program on S^2 so that we can get a feel for what's going and see how it fails on S^2.

In Section 3, we will explicitly go through the details of the surgery, assuming the Plumming lemma. In section 4, we'll see why the manifold resulting on the surgery on S^3 and $S^2 \times S^1$ will be diffeomorphic to S^3.In Section 5, we'll show how the surgery shortens distances and give a method for shrinking the diameter of S^3 to a small amount. Finally, in Section 6, we'll finish the proof by proving the Plumming lemma.

2 What happens on S^2

We first prove

Theorem 2.1. *Every metric on S^2 satisfies $Kd^2 \geq \pi^2$.*

Proof. Assume not, so that there is a metric g on S^2 with $Kd^2 < \pi^2$. Notice that this implies $d < \frac{\pi}{\sqrt{K}}$. Now, by applying the Rauch comparison theorem with (S^2, g) and the round S^2 with curvature K, we find that for a given $p \in (S^2, g)$, the first conjugate point occurs after that of the round S^2, that is, after a distance $\frac{\pi}{\sqrt{K}}$. But this says that the cut locus in TpS^2 and the conjugate locus in T_pS^2 don't intersect. But it is well known that for any metric on S^2, the cut locus and conjugate locus intersect. Thus, we have our contradiction. $\square$

Even with this in hand, we now proceed to show the surgery. So, consider two points p and $q \in S^2$ which are very far away. We want to perform surgery that doesn't alter the maximum curvature too much so that afterwords, p and q are close. To do this, fix very small open sets around p and q. Puncture them and stretch the remaining neighborhoods, as in figure 1 below.

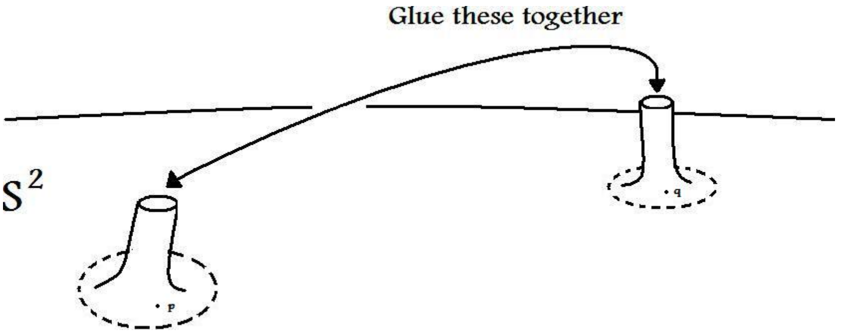

Fig 1

(While this picture file is my own creation (with microsoft paint), the picture itself is copied from Buser and Gromoll's paper)

First, it should be clear that near the boundary circles, we can make the metric flat if we like. Further, it should be clear that we can make the boundary circles have the same length. With this in hand, we can more or less obviously glue these two boundary circles together both smoothly and metrically. From this picture, we easily see several important facts: First, we created a lot of negative curvature. Thus, while we might make Kd^2 smaller, we certainly can't use this trick to make $|K|d^2$ smaller. Further, we changed the topology. It's clear the resulting manifold is diffeomorphic to a torus.

We will see that surgery will still create negative curvature on S^3, but the topology will NOT change on S^3.

3 How to metrically glue

In this section, we'll state and assume the Plumming lemma, and then using it, show how to metrically do the surgery. That is, we'll show that we can do the surgery in such a way that the metrics on each corresponding piece glue together into a smooth metric on the resulting manifold.

We'll do this for arbitrary orientable 3-manifolds.

But first, for any γ a closed geodesic, we define γ^r to be the (open) tubular neighborhood of radius r about γ.

Theorem 3.1. *(Plumming Lemma) Let γ be a simple closed geodesic on a Riemannian manifold (M^n, g) and let $r > 0$ be arbitrarily small. There exists a new metric $\tilde{g}$ on M with the following properties:*

1. *γ is a still a geodesic and the length and holonomy haven't changed.*
2. *The distances to γ haven't changed*
3. *$g = \tilde{g}$ on $M - \gamma^{4r}$*
4. *γ^{3r} is isometric to the Riemannian product $B^{3r} \times \gamma$*
5. *In $\gamma^{4r} - \gamma^{3r}$, the upper bound of the absolute curvature has increased at most by a factor C where C only depends on the dimension*

With this in mind, surgery is done as follows. Choose a closed geodesic $\gamma : S^1 \to M$ and $\tilde{\gamma} : S^1 \to N$ of length $L = l(\gamma) = l(\tilde{\gamma})$ and equal holonomy. Now, apply the Plumming lemma to both geodesics to make the manifolds flat in γ^{3r} and $\tilde{\gamma}^{3r}$. Thus, since $l(\gamma) = l(\tilde{\gamma})$, we see that γ^{3r} is isometric to $\tilde{\gamma}^{3r}$ and that these are both isometric to $B^{3r} \times \mathbb{R}/[t \to t + L]$.

In fermi coordinates in both γ^{4r} and $\tilde{\gamma}^{4r}$, the metrics looks like

$$ds^2 = dt^2 + d\rho^2 + \rho^2 d\theta^2$$

(where ρ denotes the radial distance, t is the parameter of the geodesic, and θ is the angle coordinate.)

Now, let $\phi(\rho)$ be a smooth function such that $\phi(\rho) = 2.5r$ for $r \leq \rho \leq 2.4r$, $\phi(\rho) = \rho$ for $2.6r \leq \rho \leq 3r$ and $\phi'' \geq 0$ everywhere. Consider the new metric g' (in both γ^{4r} and $\tilde{\gamma}^{4r}$) given by $ds^2 = dt^2 + d\rho^2 + \phi(\rho)^2 d\theta^2$.

Notice the following:

Since the curvature of such a metric is given by $-\frac{1}{\phi(\rho)}\frac{d^2\phi}{d\rho^2} < 0$, we see that in both γ^{3r} and $\tilde{\gamma}^{3r}$, we have added negative curvature. Further, $\gamma^{2r} - \gamma^r$ and $\tilde{\gamma}^{2r} - \tilde{\gamma}^r$ are both isometric to $[0, r] \times T$ where T is a flat torus with one factor (the θ direction) of length 2π and the other (the t direction) of length $l(\gamma) = l(\tilde{\gamma})$. Finally, because the holonomies are the same, we can glue the the boundaries of $\gamma^{2r} - \gamma^r$ and $\tilde{\gamma}^{2r} - \tilde{\gamma}^r$ so that the metric stays smooth (see figure 2)

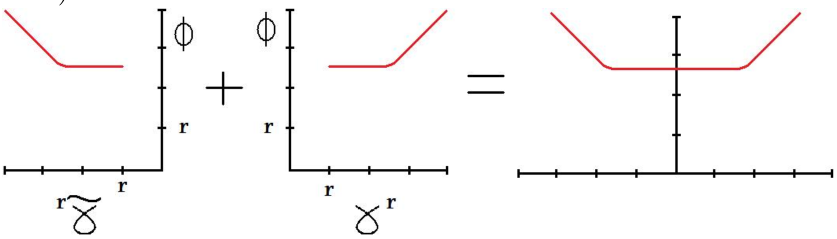

Fig 2

Thus, we see that we can do this surgery in such a way that the resulting manifold has an upper bound for curvature given by $K = C\max\{K_M, K_N\}$. (after using the Plumming lemma) had.

4 Why the surgered manifold is diffeomorphic to S^3

In this section, we show why doing this surgery on S^3 and $S^2 \times S^1$ doesn't alter the diffeomorphism class of S^3.

We first need the following:

Theorem 4.1. *Let γ be a smooth closed curve in $(S^2 \times S^1, g)$ which is isotopic to the S^1 factor. Assume that g is given as a product metric. Fix a small* *enough r, and denote the tubular neighborhood of radius r about γ as γ^r. Then the interior of the complement of γ^r, denoted X is diffeomorphic to γ^r.*

Proof. First, isotop γ so it looks like$\gamma(t) = (\text{north pole, t})$. Choose r so small that γ^r is diffeomorphic to $S^1 \times D^2$ via the diffeomorphism $exp_{\gamma(t)}(\rho\theta)$. Consider any rotation r which takes the north pole to the south pole. Finally, let $f : \gamma(t)^r \to X$ by $f(t, \rho, \theta) = (exp_{r(\gamma(t))}(g(\rho)dr(\theta)), t)$, where g is a C^∞, monotonely increasing function with $g(0) = 0$ and $g(r, \theta)$ being the smallest number such that the geodesic heading out from $r(\gamma(t))$ in the direction of θ hits γ^r.

In pictures (see Fig 3), we map the black $\gamma(t)$ to the red curve, and then the blue line, when (roughly) rotated around the red axis, sweeps out the D^2 about $r(\gamma(t))$. □

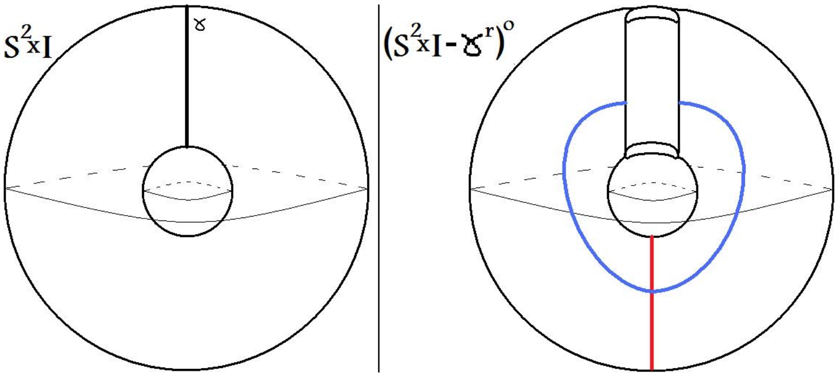

Fig 3

Now, suppose the prescribed surgery has been done on S^3 and $S^2 \times S^1$. Call the resulting manifold M. Let $j : \gamma^r \to \tilde{\gamma}^r$ be an explicit isometry. Define $h : M \to S^3$ by $f(p) = \begin{cases} p & p \in S^3 - \gamma^r \\ j^{-1}(f^{-1}(p)) & p \in S^2 \times S^1 - \tilde{\gamma}^r \end{cases}$ where f is the map defined in the previous theorem.

Then f is clearly smooth and a local diffeomorphism. Since M is compact, f is a covering map. Since S^3 is simply connected, f is a diffeomorphism.

5 Finish the proof, modulo the Plumming lemma

We have now shown that we can do a particular kind of surgery on S^3 and $S^2 \times S^1$ so that the metric varies smoothly over the new manifold, has sectional curvature bounded above by C times the maximum sectional curvatures on S^3 and $S^2 \times S^1$, and the new manifold is diffeomorphic to S^3. We'll use this to shorten distances on S^3.

First, for a given $\epsilon > 0$, Consider an rotationally symmetric almost round S^2 of curvature ≤ 1, but very close to 1. Assume wlog that the axis of rotation is a diameter spanned by the north and south poles. Fix a line of latitude which has distance $< \epsilon/2$ from the north pole and consider a geodesic $\gamma'(t)$ which begins tangent to the line of latitude. By Clairaut's theorem, this geodesic will move towards the equator, hit the corresponding latitude near the south pole, and continue bouncing around in this manor (See Figure 4 below, where a portion of the geodesic is in red). Now, it should be clear that we can adjust the shape of S^2 slightly and the line of latitude (while keeping it within $\epsilon/2$ of the north pole) so that both of the following two things occur: $\gamma'(t)$ comes within $\epsilon/2$ of every point, and $\gamma'(t)$ is a closed geodesic. For convenience, parameterize γ' so that it's domain is $[0, 1]$. It's clear that γ' has no holonomy since it's a closed geodesic on a orientable 2-manifold.

Let m be the number of times γ' hits the northern line of latitude, so that m/L is approximately 2π.

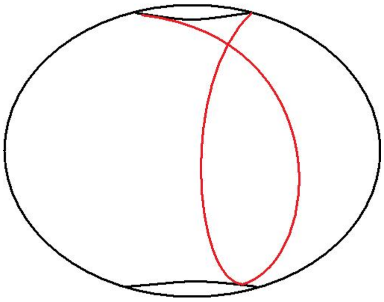

Fig 4

Now, let $S^1 = \mathbb{R}/[t \to t + \epsilon]$. Form the Riemannian product $S^2 \times S^1$. This product has curvature ≤ 1 since each piece does. Let $\tilde{\gamma}(t) = (\gamma'(t), \epsilon t)$. Notice that $\tilde{\gamma}$ comes within ϵ of any point in $S^2 \times S^1$ since if $(p, q) \in S^2 \times S^1$, then $\gamma'(t)$ comes within $\epsilon/2$ of p for some t, and at that t, ϵt is within $\epsilon/2$ of q since the diameter of S^1 is $\epsilon/2$. Further, $\tilde{\gamma}$ has no holonomy since it's a product of two 0 holonomy geodesics. Finally, specify some $r << \epsilon$ and apply the Plumming Lemma to $S^2 \times S^1$. Since we don't change radial distances, $\tilde{\gamma}$ is still ϵ dense. Also, after applying this, it's clear that $\tilde{\gamma}$ has 0 holonomy since the part in the S^2 factor in homotopic to a constant map and in a flat area, and the part in S^1 has 0 holonomy since this is true of any curve on S^1. This also raises the upper bound of the curvature to C, where C is given by the Plumming lemma.

Now, choose a round metric on S^3 so that any closed geodesic γ has $l(\gamma) = l(\tilde{\gamma}) = L$. Notice that γ obviously has no holonomy. Apply the Plumming lemma to this space (with the same r value chosen above), and notice that γ has holonomy 0, since it's a curve in a flat region and is contractible. This again raises the curvature to some number $\leq C$ (we can make L arbitrarily large by making γ' more dense).

Thus, γ and $\tilde{\gamma}$ have the same length, and the same holonomy. Thus we can perform the surgery of section 2. The resulting manifold M, which is diffeomorphic to S^3, has curvature $\leq C$.

Now I claim that there are m points $p_1, ..., p_m \in \partial\gamma^r = \partial\tilde{\gamma}^r$ such that in S^3, we have $d(p_i, p_{i+1}) \approx L/m$ and in $S^2 \times S^1$, $d(p_i, p_{i+1}) \leq 2\epsilon$. This is most easily demonstrated in the following figure (Fig 5). This follows since $d(p_i,$ north pole)$\leq \epsilon/2$ and the S^1 factor has diameter $\epsilon/2$. Further, it's clear that by choosing the joining isometry of $S^3\gamma^r$ with $S^2 \times S^1 - \tilde{\gamma}^r$, we can arrange p_1 however we want.

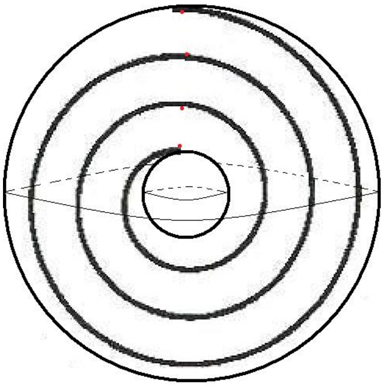

Fig 5

Now, to finish up. Let A be a finite collection on points on the round S^3 of diameter $L/2$ such that the ϵ balls around every $p \in A$ cover S^3. Let $N \in S^3$ denote the north pole. For each $p_i \in A$, there is a $q_i \in S^3$ such that both $d(p_i, q_i)$ and $d(q, N)$ are integer multiple of L/m. Now, choose two great circles γ and η such that γ comes within ϵ of p_i and q_i and η coming within ϵ of q_i and N. Choose and r so small that $r << \epsilon$, $\gamma^{4r} \cap \eta^{4r} = \emptyset$ and p_i, q_i, and $N \notin \gamma^{4r} \cup \eta^{4r}$. Do the above surgery on γ, arranging for a p_i' and q_i' to being in $\partial\gamma^r$ so that $d(p_i, p_i') \leq \epsilon$, $d(q_i, q_i') \leq \epsilon$, and $d(p_i', q_i') \leq 2\epsilon$. Next, repeat this on η, arranging for a q_i* and an $N*$ such that $d(q_i, q_i*) \leq \epsilon$, $d(N, N*) \leq \epsilon$, and $d(q_i*, N*) \leq 2\epsilon$. (See Fig 6).

We then have that $d(p_i, N) \leq d(p_i, p_i') + d(p_i', q_i') + d(q_i', q_i) + d(q_i, q_i*) + d(q_i*, N*) + d(N*, N) = (1 + 2 + 1 + 1 + 2 + 1)\epsilon = 8\epsilon$.

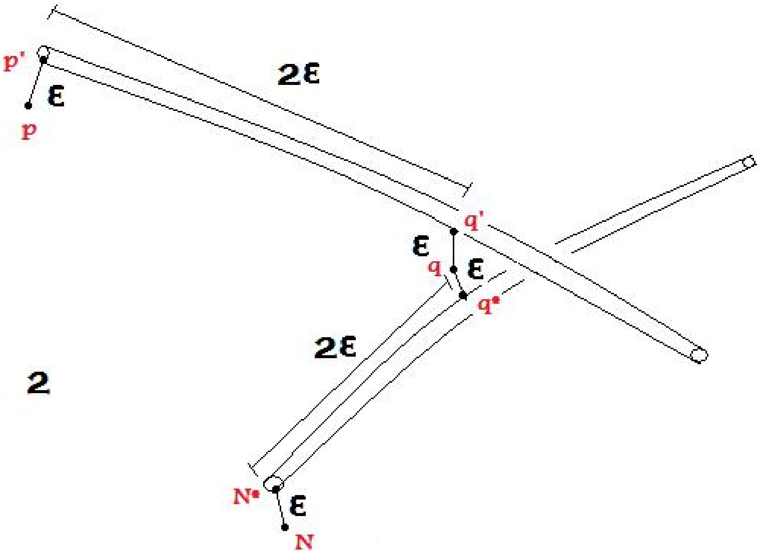

Fig 6

(Again, while I drew the picture myself, the picture can be found in Buser and Gromoll's paper)

Now, we do this for all i simultaneously, just being careful to pick r small enough that none of the γ^rs or η^rs intersect. Since every point on S^3 is within ϵ of some p_i, it follows that any point on S^3 has distance $\leq 10\epsilon$ from any other point. Further, since any point which is added by the $S^2 \times S^1 - \tilde{\gamma}^r$ is a distance $\leq \epsilon$ from some point in S^3, this gives us a diameter $\leq 12\epsilon$.

Since our upper curvature bound was C (and depended only on the dimension), we have that $Cd^2 = 144C\epsilon^2$, which can be made as small as desired.

6 The Plumming lemma

In this section we prove the Plumming lemma. Roughly, this says that around any closed geodesic, we can flatten the metric. The cost is that the is a constant $c = c(n)$ such that the maximum of the absolute values of the curvature get's multiplied by c.

Theorem 6.1. *(Plumming Lemma) Let γ be a simple closed geodesic on a Riemannian manifold (M^n, g) and let $r > 0$ be arbitrarily small. There exists a new metric $\tilde{g}$ on M with the following properties:*

1. *γ is a still a geodesic and the length and holonomy haven't changed.*
2. *The distances to γ haven't changed*
3. *$g = \tilde{g}$ on $M - \gamma^{4r}$*
4. *γ^{3r} is isometric to the Riemannian product $B^{3r} \times \gamma$*

5. In $\gamma^{4r} - \gamma^{3r}$, the upper bound of the absolute curvature has increased at most by a factor C where C only depends on the dimension

Proof. Assume r is small enough so that γ^{4r} is diffeomorphic to $S^1 \times R^{n-1}$. In this neighborhood (or rather, in a slightly smaller open set U), consider fermi coordinates. That is, choose and o.n., parallel frame $V_1(t), ..., V_n(t)$ with $V_n(t) = \dot{\gamma}(t)$, where γ is unit speed.

Define $f(x^1, ..., x^n) = exp_{\gamma(x^n)}(\sum x^i V_i)$. This gives us a coordinate map.

In these coordinates, the metric can be expressed as

$$g_{ij} = \delta_{ij} + h_{ij}$$

where

$$h_{ij} = -\frac{1}{3} R_{ikjl} x^j x^l + \text{higher order terms}$$

.

Now, let X be any unit vector in $(T_{\gamma(t)}M)^{\perp}$ and consider the map $\rho \to exp_{\gamma(t)}(\rho X)$.

In fermi coordinates we have $\frac{\partial g_{ij}}{x^k}(0) = 0$. Further, from this it follows that $\frac{\partial g_{ij}(0)}{\partial \rho} = \frac{\partial g_{ij}}{\partial x^k}\frac{\partial x^k}{\partial \rho} = 0$ (where we sum over k). Thus, since $h_{ij}(0) = 0$, we see that

$$h_{ij} \leq \kappa \rho^2 + o(\rho^2)$$

. Here, κ is a constant larger than all the curvatures (and, in particular is an upper bound for the sectional curvatures).

Further, notice $\frac{dx^k}{d\rho} \leq 1$.

Now, since $\frac{\partial h_{ij}}{x^m}(0) = 0$, and since $\frac{\partial}{\partial \rho}\frac{\partial h_{ij}}{x^m}(0) = -\frac{2}{3} R_{imjl} \frac{\partial x^l}{\partial \rho}$ (where we sum over l), we see that $|\frac{\partial h_{ij}}{x^m}(0)| \leq R_{imjl} n$, so we have that $|\frac{\partial h_{ij}}{x^m}| \leq \kappa n \rho + O(\rho^2)$.

Finally, we estimate $\frac{\partial^2 h_{ij}}{\partial x^k \partial x^l}$.

$\frac{\partial^2 h_{ij}}{\partial x^k \partial x^l}(0) = -1/3 R_{ijkl}(0) \leq \kappa$.

Thus, we see that $\frac{\partial^2 h_{ij}}{\partial x^k \partial x^l} \leq \kappa + O(\rho)$.

Now, we're ready to define $\tilde{\gamma}$. Let $\phi : [0,4] \to [0,1]$ where ϕ is a monotone smooth function, $\phi(t) = 0$ for $0 \leq t \leq 3$, $\phi(t) = 1$ for $3.5 \leq t \leq 4$, $\phi'(t) \leq 10$ and $|\phi''(t)| \leq 100$.

Define $\tilde{g}_{ij} = \delta_{ij} + h_{ij}\phi(\rho/r)$. We'll implicitly assume that r is small enough that all the o and O terms are negligible.

Now, with this new metric it's clear that 1,3, and 4 hold.

To see that 2 holds, simply notice that since we only rescale in the radial directions, the old radial geodesics are now badly parameterized geodesics. This allows us to estimate the new radial distance in terms of the old (that is, new radial distance $\leq$ D* old radial distance), which is still enough to complete the proof. However, Buser and Gromoll state that "2 is obvious"', which I don't see.

Finally, to see that 5 holds, we compute.

$$|\tilde{g}_{ij} - \delta_{ij}| = |h_{ij}\phi| \leq |h_{ij}| \leq \kappa\rho^2$$

$$|\frac{\partial \tilde{g}_{ij}}{\partial x^k}| \leq |\frac{\partial h_{ij}}{\partial x^k}| + 10/r|h_{ij}| \leq const\kappa\rho$$

. (This constant depends on the dimension)

$$|\frac{\partial^2 \tilde{g}_{ij}}{\partial x^k \partial x^l}| \leq |\frac{\partial^2 \phi}{\partial x^k \partial x^l} 1/r^2 h_{ij}| + |1/r \frac{\partial \phi}{\partial x^l}\frac{\partial h_{ij}}{\partial x^k}| + |1/r \frac{\partial \phi}{\partial x^k}\frac{\partial h_{ij}}{\partial x^l}| + |\phi \frac{\partial^2 h_{ij}}{\partial x^k \partial x^l}|$$

and this is easily seen to be

$$\leq 100\kappa + 10\kappa + 10\kappa + \kappa = 121\kappa$$

.

Since curvature is a second order term, the fact that we have estimates on $\tilde{g}$ and it's first and second order derivatives, we get an estimate on the curvature. Thus, we know that $|\tilde{R}| \leq const\kappa$, which gives us the desired result.

□

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