# Curvature of Invariant Metrics on Compact Lie Groups

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**Abstract**

We aim to give a relatively self contained proof of the following two statements. First is an easy consequence of one of O'Neill's formulae: any compact Lie group has a bi-invariant metric, and given any bi-invariant metric, the sectional curvature is non-negative. Second is a theorem first proved by Wallach: Suppose a compact Lie group G has a left invariant metric of positive sectional curvature. Then G is diffeomorphic to $S^3$ or $SO(3)$. This paper assumes a decent working knowledge of algebraic topology and Riemannian geometry.

# Contents

# 1 Introduction

In this paper, I hope to have a (relatively)[1] self contained demonstration of the following:

**Theorem 1.1.** *Let G be a compact Lie group with left invariant metric* $<,>$. *Assume that with this metric, G is positively curved. Then G is diffeomorphic to* $S^3 = SU(2)$ *or G is diffeomorphic to* $SO(3)$.

This is in sharp contrast with the following:

**Theorem 1.2.** *Let G be a compact Lie group. Then G has a bi-invariant metric. Further, with any bi-invariant metric, G has non-negative curvature.*

The proof of Theorem 1.2 is fairly straight forward and will be covered in section 3. On the other hand, the proof of Theorem 1.1 requires a bit of work, and at least at this time, requires taking three theorems on faith: that every point in a Lie group G is contained in a maximal torus, the the group exponential map is surjective onto a copmact Lie group, and any two maximal tori are conjugate. A proof of these facts can be found in Bröcker and tom Dieck's Representations of Compact Lie Groups.

The rest of the paper will be organized as follows: section 2 will contain the necessary definitions to get started, while section 3 will contain preliminary theorems regarding metrics on lie groups. Section 4 will contain a proof of Theorem 1.2 and section 5 will cover a few theorems on Killing fields in general and section 6 will be devoted to proving Theorem 1.1, first for even dimensional compact Lie groups and then for odd dimensional Lie groups. Finally, section 7 will contain the references.

[1] I assume familiarity with basic Riemannian geometry and very basic Lie Group theory. In particular, I assume a familiarity with the metric tensor and Riemann curvature tensor. Also, the long exact sequence in homotopy groups for a fiber bundle appears. For the student not entirely familiar with these, I recommend Do Carmo's Riemannian Geometry book for the geometry and Hatcher's Algebraic Topology for the algebraic topology that comes up.

# 2 Definitions and Preliminaries

**Definition 2.1.** Let $G$ be a group with multiplication $\mu : G \times G \to G$. $G$ is called a Lie group if it has a smooth structure compatible with it's algebraic structure in the sense that the multiplication map $\mu : G \times G \to G$ is smooth as is the inversion map $i : G \to G$ with $i(g) = g^{-1}$. The identity of $G$ is denoted by $e$.

**Definition 2.2.** Let $G$ be a Lie group and let $L_g : G \to G$ and $R_g : G \to G$ denote left multiplication and right multiplication respectively. That is, $L_g(p) = gp$ and $R_g(p) = pg$. We say a vector field X is left (respectively right) invariant if $L_{g*}X = X$ (respectively, $R_{g*}X = X$ for all $g \in G$). We say X is bi-invariant if it is both left and right invariant.

**Definition 2.3.** Let $G$ be a Lie group with $<,>$ a metric. The metric is said to be left invariant if $L_g^* <,>=<,>$ for all $g \in G$, where $L_g^*$ denotes the pullback. Similarly, the metric is said to be right invariant if $R_g^* <,>=<,>$ for all $g \in G$. A metric which is both left and right invariant is called bi-invariant.

**Definition 2.4.** Let X be a vector field on a Riemannian manifold (M,$<,>$). X is called a Killing vector field, or simply a Killing field, if it's local flow $\phi_t(q) = \phi(t, q)$ is an isometry for all t it's defined for.

We now include a lemma here, simply because it doesn't seem to fit anywhere else.

**Lemma 2.1.** *If $X$ is a vector field (not necessarily a Killing field), and if $\phi_t$ denotes the (local) flow of $X$ at $p$, and if $X(p) = 0$, then $\phi_t(p) = p$ for all t it's defined for.*

*Proof.* $\frac{\partial \phi_t(p)}{\partial t}|_{t=s_0} = \frac{\partial \phi_{s_0+t}(p)}{\partial t}|_{t=0} = \frac{\partial \phi_s \circ \phi_t(p)}{\partial t}|_{t=0} = d\phi_s \frac{\partial phi_t(p)}{\partial t}|_{t=0} = d\phi_s X(p) = d\phi_s 0 = 0$. Thus, the map $s \mapsto \phi_s(p)$ is constant, so, $\phi_s(p) = \phi_0(p) = p$, as claimed. □

# 3 Facts about Invariant Objects on Compact Lie Groups

The proof of Theorem 1.2 will follow from several lemmas. The big picture is as follows: First we will see that any Lie group admits a left invariant metric. After this, we'll show how to turn the left invariant metric into a bi-invariant metric. We will then derive a formula due to O'Neill which expresses the curvature of Lie Group in terms of the Lie Bracket. The result, that any compact Lie group has a bi-invariant metric with non-negative sectional curvature, will follow easily from this formula.

**Theorem 3.1.** *Given let $p \in G$ and let $v \in T_pG$ be any vector. Then there is a unique left invariant vector field $V$ such that $V(p) = v$.*

*Proof.* For any $q \in G$, define $V(q) = L_{qp^{-1}*}v$. I claim $V$ is the desired vector field. First, notice that $V(p) = L_{pp^{-1}*}v = L_{e*}v = Id_*v = v$. Further, for any $r \in G$, $L_{r*}V(q) = L_{r*}(L_{qp^{-1}}v) = L_{rqp^{-1}*}v = V(rq)$, so that $V$ is left invariant.

To see uniqueness, let $W$ be any other left invariant vector field with $W(p) = v$. Since the pushforward map is a linear map, it's clear that $W - V$ is a left invariant vector field with $(W - V)(p) = W(p) - V(p) = 0$. But then $(W - V)(q) = L_{qp^{-1}*}(W(p) - V(p)) = L_{qp^{-1}*}0 = 0$, again since the pushforward map is linear. Thus, $W - V \equiv 0$, so that $W = V$. □

**Theorem 3.2.** *Every Lie group $G$ admits a left invariant metric.*

*Proof.* Let $g_e$ be any inner product on $T_eG$. Since $T_eG$ is just a vector space, we can certainly find one. For any $p \in G$, let $g_p = L^*_{g^{-1}}g_e$. Thus, we have $g$ defined at all points. Notice that it is smooth since, by definition, G's multiplication is smooth. I claim that g is left invariant. To see this, let $h \in G$. Then $L^*_h g_p = L^*_h(L_{p^{-1}}g_e) = (L_{p^{-1}} \circ L_h)^* g_e = (L_{p^{-1}h})^* g_e = g_{h^{-1}p}$ as claimed. □

And we also need a classification of bi-invariant metrics.

**Theorem 3.3.** *A Lie group $G$ has has a bi-invariant metric iff $T_eG$ has an inner product which is conjugation invariant.*

*Proof.* Assume $g$ is bi- invariant and let $h \in H$. Then

$$C^*_h g_e = (L_h \circ R_{h^{-1}})^* g_e = R^*_{h^{-1}} L^*_h g_e = R_{h^{-1}} g_{h^{-1}} = g_e$$

Therefore, $g_e$ is conjugation invariant.

Now, assume $g_e$ is conjugation invariant at the identity and let $g_p = L^*_{p^{-1}} g_e$. Then g is left invariant by the same proof as in Theorem 3.1. Now, let $h, k \in G$. Then

$$R^*_h g_k = R^*_h L^*_{k^{-1}} g_e = R^*_h L^*_{k^{-1}} C^*_h g_e$$

by definition and conjugation invariance. But this is equal to

$$(L_{k^{-1}} \circ R_h)^* (L_h \circ R_{h^{-1}})^* g_e = (R_h \circ L_{k^{-1}})^* R_{h^{-1}} L^*_h g_e = L^*_{k^{-1}} R^*_h R_{h^{-1}} L^*_h g_e$$

since $L_h$ and $R_k$ commute for any $h$ and $k$. Now, this is equal to

$$L^*_{k^{-1}} L^*_h g_e = (L_h \circ L_{k^{-1}})^* g_e = L^*_{hk^{-1}} g_e = g_{kh^{-1}}$$

so that $g$ is also right invariant. Thus, $g$ is bi-invariant.

□

Before we prove that every compact Lie group G admits a bi-invariant metric we need to know that G has a right invariant volume form on it. This is easy to show using right translation, in an analogous fashion to the way we showed G has a left invariant metric and left invariant vector fields. However, I'm am quite fond of the following proof, which gives us a stronger result than we need.

**Theorem 3.4.** *Every compact group admits a bi- invariant volume form (in particular, it has a right invariant volume form).*

*Proof.* Let $\omega_e$ be any n covector on $T_eG$ (n = dim G). Let $\omega_p = L^*_{p^{-1}}\omega$. Then $\omega$ is left invariant, just as above. Notice briefly that $G$ has a nonvanishing volume form, so G is orientable. I claim it's also right invariant. To see this, let $g \in G$ and consider $R^*_g\omega$. First note that $R^*_g\omega$ is left invariant since for $h \in G$, $L^*_h(R^*_g\omega) = (R_g \circ L_h)^*\omega = (L_h \circ R_g)^*\omega = R^*_g L^*_h\omega = R^*_g\omega$. Since the dimension of the vector space of n-forms at any given point is 1, it follows that $R^*_g\omega = f(g)\omega$, with $f : G \to \mathbb{R}$ some smooth function. Now, since $R_{gg'} = R_g R'_g$, we have that $f(gg')\omega = R^*_{gg'}\omega = R^*_{g'} R^*_g\omega = R_{g'^*} f(g)\omega = f(g) R^*_{g'}\omega = f(g)f(g')\omega$, so that $f(gg') = f(g)f(g')$, i.e., f is a group homomorphism to $(\mathbb{R} - \{0\}, \cdot)$. Therefore, $f(G)$ is a compact, connected subgroup of $\mathbb{R}^+$. If $K$ is any nontrivial subgroup of $\mathbb{R}^+$, then $\exists k \in K$ with $k > 1$ (since every non-1 element is either bigger than 1, or its inverse is). But then $k^n \in K$, so that K is not bounded, so K is not compact. Thus, since $f(G)$ is compact, we must have $f(G) = 1$. But that means $R^*_g\omega = f(g)\omega = \omega$, so $\omega$ is right invariant as well. Therefore, $\omega$ is bi-invariant. □

With this in hand, we now prove:

**Theorem 3.5.** *If G is a compact Lie group, then G has a bi-invariant metric.*

*Proof.* We aim to show that $T_eG$ has a conjugation invariant metric $\tilde{g}_e$. By Theorem 3.2, if we define $\tilde{g}_p = L^*_{p^{-1}}\tilde{g}$, then this metric will necessarily be bi-invariant.

So, let $g_e$ by any inner product on $T_eG$. Let $\omega$ be a right invariant volume for G. Define

$$\tilde{g}_e(u,v) = \int_G C^*_h g_e(u,v)\omega(h)$$

This integral converges since $G$ is compact Then $\tilde{g}_e$ is clearly a metric, so we need only check it's conjugation invariant. So, let $k \in G$. Then

$$(C^*_k\tilde{g})(u,v) = \tilde{g}(C_{k*}u, C_{k*}v) = \int_G (C^*_h g)(C_{k*}u, C_{k*}v)\omega(h)$$

But this is equal to

$$\int_G C^*_k(C^*_h(g(u,v))\omega(h) = \int_G (C^*_{hk}(g(u,v))\omega(h)$$

Now consider $R_{k^{-1}} : G \to G$, which is clearly a diffeomorphism. Then by Fubini's theorem,

$$\int_G (C^*_{hk}g(u,v)\omega(h) = \int_G R^*_{k^1}(C^*_{hk}g(u,v))\omega(h)$$

But for fixed $h$, $C^*_{hk}g(u,v)$ is just some fixed real number, so by linearity we get that the above is

$$\int_G (C^*_{hk}g(u,v)R^*_{k^{-1}}\omega(h)$$

But $\omega$ is right invariant, so we get

$$\int_G (C^*_{hk}g(u,v)\omega(hk)$$

Now, let j = hk to get

$$\int_G (C^*_{hk}g(u,v)\omega(hk) = \int_G (C^*_j g(u,v))\omega(j) = \tilde{g}(u,v)$$

□

A nice feature of bi-invariant metrics is this:

**Theorem 3.6.** *If $g$ is a bi-invariant metric on $G$ and $X$, $Y$, and $Z$ are any 3 left invariant vector fields, then $g([X,Y],Z) = -g(X,[Y,Z])$.*

Before we prove this, we need a quick lemma:

**Lemma 3.7.** *If $\phi_t$ is the local flow of a left invariant vector field X through the identity, then the flow is given by $R_{\phi_t}$.*

*Proof.* Let $\phi_t(t)$ be the (local) flow of X through the identity. That is, $\phi_0(p) = p$ and $\frac{\partial \phi}{\partial t}|_{t=0}(p) = X(p)$. I claim that for any $g \in G$, the flow of $\phi_t$ is given by $R_{\phi_t}(g)$. To see this, notice that $X(g) = L_{g*}X(e) = L_{g*}\frac{\partial \phi_t}{\partial t} = \frac{\partial L_g \circ \phi_t}{\partial t}$, so $L_g \circ \phi_t$ is the flow through $g$. But $L_g \circ \phi_t = g\phi_t = R_{\phi_t}(g)$, so that $R_{\phi(t)}$ is the flow of X. □

With this in hand, we now prove Theorem 3.5.

*Proof.* Let $\phi_t$ is the local flow of X through the identity. By the above lemma, $R_{\phi_t}$ is the flow of X. Recall that $[Y, X]$ is defined by

$$[Y, X] = \lim_{t \to 0} \frac{1}{t}(\phi_{t*}Y - Y) = \frac{d}{dt}|_{t=0}\phi_{t*}YR$$

So if X is left invariant, we have

$$[Y, X] = \frac{d}{dt}|_{t=0}R_{\phi_t *}Y$$

Now, let X, Y, and Z be left invariant vector fields. Then $< Y, Z >=< R_{\phi_t *}L_{\phi_{-t}*})Y, R_{\phi_t *}L_{\phi_{-t}*}Z >$. But since Y and Z are left invariant, we have $< Y, Z >=< R_{\phi_t *}Y, R_{\phi_t *}Z >$. Now, differentiating both sides of this equation and setting $t = 0$ yields

$$0 = \frac{d}{dt}|_{t=0} < R_{\phi_t *}Y, R_{\phi_t *}Z >$$

By the product rule, we get that this is

$$< \frac{d}{dt}|_{t=0}\phi_{t*}Y, R_{\phi_0 *}Z > + < R_{\phi_0 *}Y, \frac{d}{dt}|_{t=0}R_{\phi_t *}Z >$$

which by definition (and using the fact that $\phi_0 = Id$) is

$$< [Y, X], Z > + < Y, [Z, X] >$$

Thus, $0 =< [X, Y], Z > + < Y, [X, Z] >$, as desired. □

# 4 A Proof of Theorem 1.2

Our goal in this section is to show that if G is a compact lie group with bi-invariant metric, and if $\sigma$ is a two plane in $T_pM$ spanned by orthonormal, left-invariant vectors $U$ and $V$, then $K(\sigma) = \frac{1}{4}||[U, V]||^2$. From here, it's clear that $G$ will have non-negative curvature, since any orthonormal $u, v$ in $T_pG$ have unique left-invariant extensions to all of G. We begin with a series of theorems which show the relationship between geodesics through the identity and one parameter subgroups (flows of left invariant vector fields through the identity).

**Theorem 4.1.** *Let $G$ be a compact lie group with bi-invariant metric, and let $\nabla$ be the Levi-Civita connection. If $X$ is a left invariant vector field, then $\nabla_X X = 0$. In particular, the flow through the identity of $X$ is a geodesic and a one parameter subgroup. Conversely, since for any geodesic $\gamma$ through the identity, there is a unique left invariant vector field $X$ such that $X(e) = \gamma'(0)$, it follows that one parameter subgroups through the identity and geodesics coincide.*

*Proof.* Let Y be any left invariant vector field. We begin with the well known formula

$$2 < \nabla_X X, Y >= X < X, Y > +X < X, Y > -Y < X, X >$$

$$+ < X, [Y, X] > + < X, [Y, X] > - < Y, [X, X] >$$

which is usually derived during the proof of the existence of the Levi-Civita connection. Since X and Y are left invariant, $< X, X >$ and $< X, Y >$ are constant. Thus, we have $2 < \nabla_X X, Y >= 2 < X, [Y, X] >$. From Theorem 3.5, we have $< X, [Y, X] >= - < X, [X, Y] >= - < [X, X], Y >= 0$. Thus, $< \nabla_X X, Y >= 0$ for any left invariant Y. Since any vector can be extended to a unique left invariant vector field, it follows that $\nabla_X X = 0$. □

This gives us a very useful formula for the covariant derivatives of left invariant vector fields.

**Lemma 4.2.** *For $X$ and $Y$ left invariant vector fields, $\nabla_X Y = \frac{1}{2}[X, Y]$*

*Proof.* We have $0 = \nabla_{X+Y} X + Y = \nabla_X X + \nabla_X Y + \nabla_Y X + \nabla_Y Y = \nabla_X Y + \nabla_Y X = \nabla_X Y + \nabla_X Y - [X, Y] = 2\nabla_X Y - [X, Y]$. Thus, we have $2\nabla_X Y = [X, Y]$, so that $\nabla_X Y = \frac{1}{2}[X, Y]$. □

This, in turn, yields a nice formula for the curvature tensor of three left invariant vector fields.

**Lemma 4.3.** *For $X, Y$, and $Z$ left invariant vector fields, we have*

$$R(X,Y)Z = \frac{1}{4}[[X,Y],Z]$$

.

*Proof.* By definition of the curvature tensor, we have

$$R(X,Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z$$

Applying lemma 4.2 one times shows this is equal to

$$\nabla_Y \frac{1}{2}[X,Z] - \nabla_X \frac{1}{2}[Y,Z] + \frac{1}{2}[[X,Y],Z]$$

Applying lemma 4.2 again, we have that this is equal to

$$\frac{1}{4}[Y,[X,Z]] - \frac{1}{4}[X,[Y,Z]] + \frac{1}{2}[[X,Y],Z]$$

But, from the Jacobi identity, we have

$$[X,[Y,Z]] + [Z,[X,Y]] + [Y,[Z,X]] = 0$$

so that

$$[[X,Y],Z] = [X,[Y,Z]] - [Y,[X,Z]]$$

Plugging this into the above yields

$$R(X,Y)Z = -\frac{1}{4}[[X,Y],Z] + \frac{1}{2}[[X,Y],Z] = \frac{1}{4}[[X,Y],Z]$$

as claimed. □

We are now ready to establish the promised formula.

**Theorem 4.4.** *Let $g \in G$, with $G$ a compact Lie group with bi-invariant metric. Let $u, v \in T_g G$ be orthonormal. Let $\sigma = span\{u, v\}$. Let $U$ and $V$ denote the unique extensions of $u$ and $v$ to left invariant vector fields. Then $K(\sigma) = \frac{1}{4}||[U,V]||^2$.*

*Proof.*

$$K(\sigma) = < R(U,V)U, V > = < \frac{1}{4}[[U,V],U], V >$$

by the definition of sectional curvature, and Lemma 4.3. This, in turn is equal to

$$\frac{1}{4} < [U,V],[U,V] > = \frac{1}{4}||[U,V]||^2$$

where we have used Theorem 3.5 to move the Lie bracket around. □

## 5 Facts about Killing Fields

We now switch gears and work towards proving Theorem 1.1 Our primary tool for this proof will consist of some facts about killing fields, and as well as a handful of lemmas coming from basic representation theory (some of which we'll prove). Most (all?) of the following theorems on Killing fields are given as problems in Do Carmo.

We begin with a characterization of Killing fields:

**Theorem 5.1.** *The following are equivalent:*

1. *X is a killing field*
2. $< \nabla_Y X, Z > + < \nabla_Z X, Y >= 0$ *for any vector fields* $Y$ *and* $Z$
3. $L_X g = 0$ *where* $L_X$ *denotes the Lie derivative.*

*Proof.* ($1 \Rightarrow 3$)Assume X is a killing field, and let $\phi_t$ be the flow of X. Then $\phi_t^* g = g$, so $\phi_t^* g - g = 0$ so $\lim_{t\to 0} \frac{1}{t}(\phi_t^* g - g) = 0$. But this is the definition of Lie derivative, so $L_X g = 0$.

($3 \Rightarrow 1$). Conversely, assume $L_X g = 0$. Thus,

$$0 = \lim_{t\to} \frac{1}{t}(\phi_t^* g - g)$$

Now, let $s \in \mathbb{R}$ with $\phi_{t+s} = \phi_{s+t}$ defined. Thus, we have

$$0 = \lim_{t\to 0} \frac{1}{t}(\phi_t^* g - g) = \lim_{t\to 0} \frac{1}{t}(\phi_t^* \phi_s^* g - \phi_s^* g)$$

But this implies that the map which sends

$$s \to < \phi_{s*} v, \phi_{s*} v >$$

is constant for any s, so

$$< \phi_{s*} v, \phi_{s*} v >=< \phi_{0*} v, \phi_{0*} v >=< v, v >$$

for any v. Thus, $\phi_s$ acts by isometries for all s. Thus, X is a killing field.

($2 \Rightarrow 3$)Assume $< \nabla_Y X, Z > + < \nabla_Z X, Y >= 0$. Since

$$\nabla_Y X - \nabla_X Y = [Y, X]$$

we have

$$< [Y, X], Z > + < [Z, X], Y > + < \nabla_X Y, Z > + < \nabla_X Z, Y >= 0$$

so that

$$< [Y, X], Z > + < [Z, X], Y > + X < Y, Z >= 0$$

But this is just an equivalent formulation of $L_X(<,>)(Y,Z)$. Thus, since Y and Z were arbitrary, we have $L_X <,>= 0$.

($3 \Rightarrow 2$)First note that by continuity it suffices to prove the theorem at points p with $X(p) \neq 0$. Let $V \subseteq T_pM$ with $V = X(p)^{\perp}$. Let $B \subseteq T_pM$ such that $exp_p$ is defined on B. Let $S = exp_p(V \cap B)$. Then S is a codimension 1 submanifold which is orthogonal to $X(p)$. Let S have slice coordinates $x_1, ..x_{n-1}$. Then, for small t, $(x_1, .., x_{n-1}, t)$ forms coordinates for the whole manifold. Further, we can choose t (by rescaling) so that $\frac{\partial}{\partial t}|_p = X(p)$.

Then

$$< \nabla_{\frac{\partial}{\partial x_j}} X, \frac{\partial}{\partial x_i} > + < \nabla_{\frac{\partial}{\partial x_i}} X, \frac{\partial}{\partial x_j} >$$

$$=< [\frac{\partial}{\partial x_j}, X], \frac{\partial}{\partial x_i} > + < \nabla_X \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_i} > + < [\frac{\partial}{\partial x_i}, X], \frac{\partial}{\partial x_j} > + < \nabla_X \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} >$$

by using the equation $\nabla_X Y - \nabla_Y X = [X, Y]$. But coordinate vector fields have 0 lie bracket. Thus, this equation reduces to

$$< \nabla_{\frac{\partial}{\partial x_j}} X, \frac{\partial}{\partial x_i} > + < \nabla_{\frac{\partial}{\partial x_i}} X, \frac{\partial}{\partial x_j} >=< \nabla_X \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_i} > + < \nabla_X \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} >$$

and this is equal, in turn, to

$$X < \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} >$$

But then

$$X < \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} >= X < \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} > + < [X, \frac{\partial}{\partial x_j}], \frac{\partial}{\partial x_i} > + < [X, \frac{\partial}{\partial x_i}], \frac{\partial}{\partial x_i} >$$

which is equal to

$$(L_X <,>)(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$$

But

$$L_X <,>= 0$$

by assumption. Thus,

$$< \nabla_{\frac{\partial}{\partial x_j}} X, \frac{\partial}{\partial x_i} > + < \nabla_{\frac{\partial}{\partial x_i}} X, \frac{\partial}{\partial x_j} >= X < \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} >= 0$$

The final result follows by linearity.

□

Our goal now is to prove that if we have a compact manifold M of even dimension with positive curvature, then any killing field must have a 0.

**Definition 5.1.** Let X be a killing field and let $p \in M$. Then we define the map $A_X : T_pM \to T_pM$ by $A_X(y) = \nabla_y X$. This is well defined because the covariant derivative is tensorial in the bottom coordinate.

**Lemma 5.2.** *(Berger). Suppose X is a killing field on a compact Riemannian manifold M. Let $f : M \to \mathbb{R}$ be given by $p \to < X(p), X(p) >$. Let $p \in M$ be a critical point of $f$. Then, we have for any vector field V that $< A_X(V(p)), A_X(V(p)) >= \frac{1}{2}V(p)(V(f))+ < R(X,V)X,V >$ and that $< A_X V(p), X > = 0$.*

*Proof.* We do the second claim first. Since p is a critical point of $f$, we have for any vector field V,

$$0 = V(f) = V < X, X >= 2 < \nabla_V X, X >$$

so

$$0 =< \nabla_V X, X >=< A_X(V), X >$$

as claimed.

The proof of the first claim is a lengthy computation. First, notice that $\nabla_X X(p) = 0$. This follows since

$$< \nabla_X X, Z >= - < \nabla_Z X, X >= -1/2Z < X, X >= 0$$

since p is a critical point. Also, note that in general, $< \nabla_Z X, Z >= 0$ since $< \nabla_Z X, Z >= - < \nabla_Z X, Z >$ from one of the equivalent notions of Killing field from Theorem 5.1

Then we have

$$\frac{1}{2}V_p(V < X, X >)+ < R(X,V)X,V >$$

$$= \frac{1}{2}V(< \nabla_V X, X > + < X, \nabla_V X >)+ < \nabla_V \nabla_X X, V >$$
$$- < \nabla_X \nabla_V X, Z > + < \nabla_{[X,V]} X, V >$$
$$= V < \nabla_V X, X > +V < \nabla_X X, V > - < \nabla_X X, \nabla_V V >$$
$$- < \nabla_X \nabla_V X, V > + < \nabla_{[X,V]} X, V >$$
$$= V < \nabla_V X, X > -V < \nabla_V X, X > - < \nabla_X X, \nabla_V V >$$
$$- < \nabla_X \nabla_V X, V > + < \nabla_{[X,V]} X, V >$$

$$= - < \nabla_X \nabla_V X, V > + < \nabla_{[X,V]} X, V >$$
$$= -X < \nabla_V X, V > + < \nabla_V X, \nabla_X V > - < \nabla_V X, [X,V] >$$
$$=< \nabla_V X, \nabla_X V > - < \nabla_V X, [X,V] >$$
$$=< \nabla_V X, \nabla_X V > - < \nabla_V V, \nabla_X V - \nabla_V X >$$
$$=< \nabla_V X, \nabla_V X >=< A_X(V), A_X(V) >$$

as claimed.

□

Finally, we need one quick lemma:

**Lemma 5.3.** *Let $V$ be a vector space over $\mathbb{R}$ with inner product and let $f : V \to V$ be an antisymmetric isomorphism, i.e., an isomorphism satisfying $< f(v), w >= - < f(w), v >$. Then $\dim V$ is even.*

*Proof.* Suppose $\dim V$ is odd. Then $f$ has an eigenvalue $\lambda$ corresponding to an eigenvector $v \neq 0 \in V$. But then we have $\lambda < v, v >=< \lambda v, v >=< f(v), v >=< -f(v), v >$, so that $\lambda$ is 0. But then $v \in \ker f$, contradicting the fact that $f$ is an isomorphism.

□

With all these prerequisite lemmas in hand, we're ready for:

**Theorem 5.4.** *Let M be a compact manifold with positive curvature and assume it has a Killing field on it which is nonzero everywhere. Then the dimension of M is odd. Contrapostively, if M has even dimension, then every Killing field on M has a 0.*

*Proof.* Let X be a killing field on M without a zero. Let $f : M \to \mathbb{R}$ be the above map, $p \to< X(p), X(p) >$. Assume $p$ is the absolute minimum of $f$. Let $E \subseteq T_pM$ the the vector space which is orthogonal to $X(p)$, so that $\dim E = \dim M - 1$. Consider the map $A_X : E \to T_pM$. We claim that $A_X$ is an antisymmetric isomorphism from $E$ to $E$, so that $E$ has even dimension. To see this, notice first off that we calculated above that

$$< A_X(Z), X >=< \nabla_Z X, X >= - < \nabla_Z X, X >= 0$$

so that $im(A_X) \subseteq E$. It's antisymmetric since

$$< A_X(Z), W >=< \nabla_Z X, W >= - < \nabla_W X, Z >= - < A_X(W), Z >$$

Thus, we need only show it's an isomorphism. Since the dimension of the domain and range are the same, we need only show it's 1-1. So, let

$Z \in T_pM$ with $Z \neq 0$. Then $< A_X(Z), A_X(Z) >= Z(Z(< X, X >)) + < R(X,Z)X, Z >$. Since p is the absolute minimum, $Z(Z < X, X >)(p) \geq 0$. Since M has positive curvature, $< R(X,Z)X, Z >> 0$. Thus, $< A_X(Z), A_X(Z) >> 0$, so that $A_X$ maps nonzero vectors to nonzero vectors. Thus, it's 1-1 and so $\dim E$ is even. Hence, $\dim M$ is odd.

□

*Note* 5.1. One can view Theorem 5.4 as an infinitesimal version of the Synge-Weinstein Theorem, which states that if M is a compact, even dimensional Riemannian manifold with positive sectional curvature, then any orientation preserving isometry has a fixed point. Notice that the flow of a killing field is homotopic to the identity map (the flow IS the homotopy), so is orientation preserving.

# 6 The Proof of Theorem 1.1

## 6.1 Even dimensional case

This is a quick corollary of the work we've done thus far - we just need a quick lemma.

**Lemma 6.1.** *If G is a Lie group with left invariant metric, then a right invariant vector field X is a Killing field.*

*Proof.* Let $\phi_t$ be the flow of X through the identity. Copying the proof of Lemma 3.6 (swapping all uses of 'right' and 'left'), we find the flow of $\phi_t$ $= L_{\phi_t}$. But $L^*_{\phi_t} g = g$ since g is left invariant. Thus, the flows of X are isometries, so that X is a Killing field.

□

**Theorem 6.2.** *If G is a compact lie group of even dimension with left invariant metric g, then G does NOT have positive curvature everywhere.*

*Proof.* Assume that G HAS positive curvature. Let $X$ be a right invariant vector field on G which is nonzero everywhere. These certainly exist: any tangent vector has a unique extension to a right invariant vector field. If we start with the 0 vector at some point, then, by uniqueness, its only extension to a right invariant vector field is the 0 field. Thus, if we pick any nonzero vector at any point and extend it to a right invariant vector field, it must necessarily be nonzero everywhere.

But a right invariant vector field is a killing field by Lemma 6.1. By theorem 5.4, a Killing field on G must have a 0. Thus, we have a contradiction, so that G doesn't have positive curvature.

□

## 6.2 Odd Dimensional Case

This section requires a few more difficult lemmas, most which we state and prove below. The proofs are taken from Bröcker and tom Dieck's Representations of Compact Lie Groups. However, the following two theorems will be assumed, but not proven (until, perhaps, a later draft).

**Theorem 6.3.** *(Conjugation Theorem) Let G be a compact Lie group. Then any 2 maximal tori ( = compact abelian subgroups) are conjugate (and thus, have the same dimension). Further, for any element $g \in G$, there is a maximal torus containing g.*

**Theorem 6.4.** *Let $G$ be a compact Lie group. Then the group exponential map is surjective.*

**Lemma 6.5.** *Let $G$ be a compact Lie group with* $\dim G > 2$. *Assume $G$ has rank 1 (that is, the maximal torus in $G$ is a copy of the circle $S^1$. Then* $\dim G = 3$.

*Proof.* Choose an Ad-invariant metric (i.e., a conjugation invariant metric) on $\mathfrak{g}$, the Lie algebra of G. Let $\mathfrak{t}$ = the Lie algebra of T. Let $H \in \mathfrak{t}$ with $|H| = 1$. Define $f : G \to S^{n-1} \subseteq \mathfrak{g}$ by $g \to Ad(g)H = C_{g*}H$. Notice that for $t \in T$, we have $f(gt) = Ad(gt)H = Ad(g)Ad(t)H = Ad(g)H$ since $H \in \mathfrak{t} \Rightarrow C_{t*}H = H$ since $C_{t*}|\mathfrak{t} = id|_{\mathfrak{t}}$. Thus, f factors through a map $\phi : G/T \to S^{n-1}$. That is, there is a smooth map $\phi$ such that $\phi \circ \pi = f$, where $\pi : G \to G/T$ is the natural projection.

We claim that $\phi$ is actually a diffeomorphism. We will first show it's injective and then that it's has constant rank. This will force $\phi$ to also be surjective and thus bijective. But a bijective map of constant rank is a diffeomorphism, so we'll be done.

To see that $\phi$ is injective, suppose $Ad(g)H = Ad(k)H$, for some $g, k \in G$. Then $Ad(gk^{-1})H = H$, so $Ad(gk^{-1})|_{\mathfrak{t}} = id|_{\mathfrak{t}}$. It follows then that $T$ is fixed pointwise by conjugation with $gk^{-1}$. Thus, $gk^{-1}t = tgk^{-1}$, so that $gk^{-1} \in Z(T)$, where $Z(T)$ denotes the center of T. But center is an abelian subgroup containing T. By maximality of T, Z(T) = T. Thus, $gk^{-1} \in T$, so that $gT = kT$, so $\phi$ is injective.

To see that $\phi$ is a diffeomorphism, first notice that $G$ acts transitively on $G/T$ by left multiplication and G acts on $S^{n-1}$ via f. Then $\phi$ is equivariant with respect to these 2 actions. That is, $\phi \circ g = f(g) \circ \phi$. This is because $(\phi \circ g)(hT) = \phi(g(hT)) = \phi((gh)T) = \phi(\pi(gh)) = f(gh) = f(g)f(h) = f(g)\phi(\pi(h)) = f(g)\phi(hT)$. Now I claim that this implies $\phi$ has constant rank. To see this, let $pT$ and $hT$ be two points in $G/T$. Choose $g \in G$ such that $g(hT) = pT$. Such a g exists since G acts transitively. But $\phi \circ g = f(g) \circ \phi$, so since g and f(g) are invertible actions, $g_*$ and $f(g)_*$ must also be invertible. Hence, we have $f(g)_*\phi_* = \phi_* g_*$, so that each of the $\phi_*$ has the same rank. But the first $\phi_*$ has domain $T_{hT}M$ while the second has domain $T_{g(hT)}M = T_{pT}M$. Thus, the rank at every point is the same, so $\phi$ has constant rank. Notice that since $\phi$ is injective and has constant rank, it follows that $\phi$ is an immersion. But since dim $G/T$ = dim $S^{n-1}$, an immersion is also a submersion. Hence, $\phi$ is an open map and so $\phi(G/T)$ is open. Since $G/T$ is compact, $\phi(G/T)$ is also closed, so that $\phi$ is surjective. Thus $\phi$ is a bijective. Hence, $\phi$ is a diffeomorphism as claimed.

With this in hand, notice we have a fiber bundle given by $S^1 = T \to$

$G \to G/T = S^{n-1}$. Then this gives rise the long exact sequence in homotopy

$$... \to \pi_2(S^{n-1}) \to \pi_1(T) \xrightarrow{i_*} \pi_1(G) \xrightarrow{\pi_*} \pi_1(S^{n-1}) \to ...$$

Now, assume for a contradiction that dim G $> 3$. Then $\pi_2(S^{n-1}) = \pi_1(S^{n-1} = 0$, so that $i_* : \pi_1(T) \to \pi_1(G)$ must be an isomorphism. Thus, we conclude that $\pi_1(G) = \mathbb{Z}$. We claim that this is a contradiction. Thus, since we assumed $n > 2$, it follows that $n = 3$ as claimed.

To see why this is a contradiction, use $i*$ to identify $\mathbb{Z} = \pi_1(T)$ and $\pi_1(G)$. Since $\phi$ is surjective and $|-H| = |H| = 1$, there is a $g \in G$ with $Ad(g)H = -H$. With $exp(sH)$ a closed curve generating $\pi_1(T)$, it follows that $C_{g*}[exp(sH)] = -[exp(sH)]$, so that $C_{g*} = -Id$. But G is connected, so there is a path $\gamma$ with $\gamma(0) = e$ and $\gamma(1) = g$, giving a homotopy between $C_e$ and $C_g$. Thus $C_{e*} = C_{g*}$ as maps on $\pi_1(T)$. But $C_{e*} = Id$. Thus, we have $-Id = C_{g*} = C_{e*} = Id$, giving us our contradiction and completing the proof of the theorem. □

**Lemma 6.6.** *Let G be a compact Lie group of rank 1 (so by Lemma 6.2* $\dim G = 3$*) The G covers* $SO(3)$*. In particular, since* $\pi_1(SO(3)) = \frac{\mathbb{Z}}{2\mathbb{Z}}$*,* $SO(3)$ *has precisely 2 connected covers - itself, and it's universal/double cover* $S^3 = SU(2)$*, G must be diffeomorphic to one of these.*

*Proof.* Define $Ad : G \to Iso(T_eG)$ as follows (we freely make use of the identification of the left invariant vector fields and $T_eG$): Choose an Ad-invariant metric on the Lie algebra $\mathfrak{g}$ of G (i.e., a metric on $\mathfrak{g} = T_eG$ which is invariant under $C_{g*}$. Thus, each $C_g$ acts by isometries on $\mathfrak{g}$, fixing the origin. Since $\mathfrak{g}$ is isomorphic (as a vector space with inner product) to $\mathbb{R}^3$ with the usual metric, we can view $Ad : G \to Iso(\mathfrak{g}) = Iso(R^3) = SO(3)$.

First, I claim that $Ad$ is an immersion- that is $d_pAd$ is injective for each $p \in G$. To see this, first recall that we can first identify $T_pG$ with $T_eG = \mathfrak{g}$ via left translation. Now, if $Ad$ wasn't an immersion, then there would be some $p \in G$ and some $x \in T_pG$ with $d_pAd(x) = 0$. Now, extend x to a nonzero left invariant vector field $X \in \mathfrak{g}$. Then, $0 = d_pAd(x) = ad(X)$. But then for any $Y \in \mathfrak{g}$, $0 = ad(X)(Y) = [X, Y]$. Choose a particular $Y$ and let $V = \text{span}\{X, Y\}$. Then this is clearly an abelian subalgebra of $\mathfrak{g}$, contradicting the fact that G has rank 1. Thus, $Ad$ is an immersion. Since dim G = dim SO(3) = 3, it follows that an immersion must also be a submersion. But submersions are open maps and so the image of Ad is open (and closed, since G is compact). Thus, the image of Ad must be all of SO(3). Further, since dim G = dim SO(3), Ad is actually a local diffeomorphism. But a local diffeomorphism whose domain is compact is a covering map.[2] □

[2]This last statement can be proved, for example, by putting any complete Riemannian

With these lemmas in hand, notice that in order to prove Theorem 1.1, we need only show:

**Theorem 6.7.** *Assume G is a compact Lie group of odd dimension and* $\dim G > 2$. *Assume G has a left invariant metric, and with this metric, G has positive curvature. Then G has rank 1.*

*Proof.* (Wallach) Let T be a one dimensional torus in G. Let $h \in T$ with $h^n$ dense in T. Then by the conjugation theorem, there is a maximal torus $M_h$ which contains h. But tori are closed, so $T \subseteq \overline{\{h^n | n \in \mathbb{Z}\}} \subseteq M_h$. Let C(T) denote the centralizer of T. That is, C(T) = $\{g \in G$ such that $gt = tg$ $\forall t \in T\}$. It is clear then that $T \subseteq M_h \subseteq C(T)$. Thus, if we can show $C(T) \subseteq T$, we'll have $T \subseteq M_h \subseteq C(T) \subseteq T$ so that $T = M_h$, then G has rank 1.

To this end, let T act on G by left multiplication and let $M = T/G$ be the space of right cosets. Notice that since G has odd dimension and dim T = 1, M has even dimension. Since T acts by isometries, M inherits a metric so that $\pi : G \to M$ is a Riemannian submersion. By an easy calculation, the curvature of $M$ is greater than or equal to the curvature of $G$, so is also positive everywhere.

Note that C(T) acts on M as follows. For any $k \in C(T)$, let k(Tg) = T(kg). In other words, find a preimage of g, multiply on the left with k, and project back down. This is well defined since if $g_1$ and $g_2$ are two lifts of g, we have $g_1 = tg_2$ for some $t \in T$ so that $kg_1 = ktg_2 = tkg_2$ so that $\pi(kg_1) = \pi(kg_2)$. Thus, the Lie algebra $\mathfrak{c}(\mathfrak{t})$ of C(T) acts as Killing vector fields on M. Thus, by Theorem 5.4, each of these has a 0.

But notice that $C(T)$ is closed - if $a_n \in C(T)$ limit to $a \in G$, then we have for any $t \in T$, $at = \lim_n a_n t = \lim_n t a_n = t \lim_n a_n = ta$, so that $a \in C(T)$ (and we have used the continuity of group multiplication to move the limit around). Thus, C(T) is compact, being a closed subset of G, which is compact. [3]Thus, the group exponential map from $\mathfrak{c}(\mathfrak{t})$ to C(T) is surjective.

Now, let X be any element of the Lie algebra of C(T). Then, X has a 0, say, at $Tg$. Thus, $0 = X(Tg) = \frac{d}{ds}|_{s=0}(exp(sX)(Tg))$. It follows (see Lemma 2.1) that $\frac{d}{ds}|_{s=s_0}(exp(sX)(Tg)) = 0$ for any $s_0$. Thus, we have $exp(sX)(Tg) = Tg$ for any s, so, by right multiplication with $g^{-1}$ we have $exp(sX)T = T$. But this means there is some $t, t' \in T$ with $exp(sX)t = t' \Rightarrow exp(sX) = t't^{-1} \in T$. Since exp is surjective onto C(T), it follows that $C(T) \subseteq T$, establishing the claim. □

metric on SO(3) and pulling it back to a metric on G. Thus, Ad would be a local isometry. Now, mimicking the proof of Hadamard's theorem (found in Do Carmo), we find that Ad is a covering map.

[3]I need to show C(T) is connected to complete this argument. I may do this someday.

# 7 References

I'll figure out how to do reference properly in tex, then do them.