

**241 Exam 1 - Due Monday June 5th at the start of class.**

**1** Compute the Fourier Series of  $f(x) = \sin^4 x$  on  $(-\pi, \pi)$ . Feel free to use any tricks to simplify computations.

**2** The equation  $-\frac{\tau}{\rho}u''(x) = \lambda u(x)$  with the constraints  $u(0) = u(l) = 0$  describes the behavior of a plucked string. Here  $\tau$  is the tension,  $\rho$  is the density, and  $l$  is the string's natural length, all constants. Find all the eigenvalues  $\lambda_1 < \lambda_2 < \dots$ . The first non-zero eigenvalue is the pitch of the string; show that you can increase the pitch by either increasing the tension, decreasing the length, or decreasing the density.

**3** Expand  $f(x) = x^4$  in a Legendre Series on  $(-1, 1)$ . How many coefficients do you need to compute and why?

**4** Do problem 19 from 12.6.2. *Hint: If stuck, I suggest re-reading the related calculations for Fourier Series in section 12.3. Then re-read the end of 5.3 to remind yourself of some standard properties of the Legendre polynomials.*

**5** For each positive integer  $n$  put  $f_n(x) = n$  on the interval  $(-\frac{1}{2n}, \frac{1}{2n})$  and  $f_n(x) = 0$  outside of the interval  $(-\frac{1}{2n}, \frac{1}{2n})$ . Draw the graph of the function for  $n = 1, 2, 3$  and 4 on a single plot, and describe what happens as  $n \rightarrow \infty$ . Now, compute the (complex) Fourier Transform of  $f_n(x)$ . Draw the Fourier Transforms of  $f_1(x)$ ,  $f_2(x)$ ,  $f_3(x)$ , and  $f_4(x)$  on a single plot and describe what happens as  $n \rightarrow \infty$ .

**6** Compute the (complex) Fourier Transform of  $f(x) = e^{-x^2}$ . *Hint: Complete the square and make use of the famous integral  $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$ .*

**7** Do 13.5.4 from the book.

**8** Parabolic coordinates  $f, h$  are defined from cartesian coordinates in the plane as follows:  $x = f - h$ ,  $y = 2\sqrt{fh}$  where  $f$  and  $h$  are never both negative. Draw pictures of some curves corresponding to constant  $f$  in the  $x - y$  plane<sup>1</sup>, and do the same for a few values for constant  $h$ . Let  $u(x, y)$  be a smooth function of  $x$  and  $y$ . Express the Laplacian  $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$  in plane-parabolic coordinates; your answer might be messy, but that's ok.

**9** Here's a different kind of PDE<sup>2</sup>. It's sometimes useful to search out a "square root" of the Laplacian. That is, to find an operator  $\partial$  such that  $\partial(\partial u) = \nabla^2 u = u_{xx} + u_{yy}$ . Suppose  $\partial$  is linear and in particular  $\partial = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}$  for some constants  $a$  and  $b$ . Show that in general  $\partial(\partial u) \neq \nabla^2 u$  for any numbers  $a$  and  $b$ . Show however that if  $a$  and  $b$  are allowed

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<sup>1</sup>For example, for polar case curves of constant  $r$  are circles and curves of constant  $\theta$  are radial lines

<sup>2</sup>equations of this type are used to describe certain quantum particles

to be *matrices* we may have  $\partial(\partial u) = \nabla^2$ . *Hint: I'll save you guesswork and go ahead and take one of  $a$  or  $b$  to be  $2 \times 2$  diagonal: then find the other. Remember that for matrices  $AB$  isn't necessarily  $BA$ .*

**10** Let  $V = \mathbb{R}^n$  be usual  $n$ -dimensional Euclidean space. We know the continuous real-valued functions  $C(V)$  on  $V$  are an infinite dimensional vector space. However, if we restrict our attention to certain types of functions things may be better behaved. Let  $V^*$  be the collection of all *linear* continuous real-valued functions on  $V$ . It is not hard to see that  $V^*$  is a vector subspace of  $C(V)$ . Exhibit a basis and compute its dimension. *Hint: Drawing picture probably won't help here. Instead, fix a standard basis  $e_1, \dots, e_n$  for  $V$  and try to describe functions in  $V^*$  in terms of how they act on this basis. Use linearity.*