Math 241 Summer '05 Exam 2

Name: ________________________________

Problem 1 - 30 Points.

a) Find all complex numbers z such that $\sin z = i$.

c) Evaluate $\oint_{|z|=1} z^n dz$ where n is any integer (your answer may depend on the choice of n). Justify your reasoning.

c) Put $f(z) = \cot z$. What is the nature of the singularity at $z = 0$? If it's a pole or an essential singularity, compute the residue.

d) Let z_0 be any point in the complex plane. Give an example of a function $f(z)$ with a removable singularity at $z = z_0$. Repeat for a pole of order m and and essential singularity.

e) Let H be the region in the complex plane consisting of all points with real part greater than 1. Does $\log z$ have an analytic branch defined on this region? Why or why not? What about $\log(z^2 - 4)$?

f) Let $f(z)$ be any analytic function on some region U of the complex plane. Let $f(z) = u(x, y) + iv(x, y)$ be its respective real and imaginary components at any point. Show That $u(x, y)$ is Harmonic. You may assume that u and v are each continuous and have partial derivatives of all orders.

Problem 2 - 15 Points.

Express the Cauchy-Riemann equations in polar coordinates. In particular, show that they can be brought into the form $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$, $\frac{\partial v}{\partial r} = \frac{-1}{r}\frac{\partial u}{\partial \theta}$ where $f(z) = u(x, y) + iv(x, y)$. You may assume that all partial derivatives of u and v are continuous if it simplifies matters.

Problem 3 - 15 Points.

Let $a > 1$ be some real number. Show that

$$\int_0^{2\pi} \frac{d\theta}{a + \cos\theta} = \frac{\pi}{\sqrt{a^2 - 1}} \tag{1}$$

Problem 4 - 15 Points.

Show that

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2} \tag{2}$$

You may assume the integral converges.

Bonus: Show that $\int_0^\infty \frac{|\sin x|}{x}\,dx$ diverges. That is, the integral above is conditionally convergent. Note that contour integration may not help much with this part.

Problem 5 - 25 Points. Formal proofs not required - just convincing arguments.

a) Let $f(z)$ be analytic near the point $z = z_0$. Show that $\int_\gamma \frac{f(w)}{w-z_0}\,dw = 2\pi i f(z_0)$ for γ any simple, closed, positively oriented contour enclosing z_0 but no other singularities of $f(z)$ - that is, prove the Cauchy Integral Formula.
Hint: It might be helpful to note that $\frac{f(w)}{w-z_0} = \frac{f(w)-f(z_0)+f(z_0)}{w-z_0}$.

b) Give the analogous formula for $f^{(n)}(z_0)$, the n'th derivative at z_0. No proof required - just state it.
Hint: If you forget the formula, differentiating under the integral sign might help you remember it.

c) Suppose $f(z)$ is analytic on the disc of radius R about z_0 and suppose also that $|f(z)| \leq M$ on the disc - i.e. $f(z)$ is bounded by M on the disc. Justify the Cauchy Estimate: $|f^{(n)}(z_0)| \leq \frac{n!M}{R^n}$.
Hint: Use your result from part *b)* and estimate.

d) Prove Liouville's Theorem: a bounded entire (i.e. analytic everywhere on the complex plane) function is constant.
Hint: Take $n = 1$ in part *c)* and take a limit. Remember z_0 can be *any* complex number where $f(z)$ is analytic.

e) Let $p(z) = a_n z^n + ... + a_1 z + a_z$ be a polynomial of degree $n > 0$ (so $p(z)$ is non-constant). Show that $p(z)$ has a complex zero.
Hint: Suppose not. Consider $\frac{1}{p(z)}$ and obtain a contradiction.

f) Let $\mathbf{A}$ be a $n \times n$ matrix. Show that $\mathbf{A}$ has a (possibly complex) eigenvalue.
Hint: Consider the characteristic polynomial of $\mathbf{A}$.

Bonus: Let G be some region in the complex plane and suppose $f(z)$ is analytic on G. Suppose there exists some point a in G such that $|f(z)| < |f(a)|$ for every z in G. Prove that $f(z)$ is constant.
This result is sometimes the "Maximum Principle" because it states that an analytic function cannot achieve its maximun on the inside of a region - it must grow larger and larger in absolute value as you approach the boundary.
Hint: Consider the Cauchy Integral Formula in polar form (i.e. the mean-value form of the Cauchy Integral Formula) and estimate.