241 Homework 2 - Due Monday May 29 at the beginning of class.

Sturm-Liouville Problems

a) Do *12.5.2* and *12.5.10*. Note that instead of a "CAS", any graphing calculator will suffice. Also, just finding approximations for the first two eigenvalues is fine.

b) *12.5.4*. First, do the problem as stated. After you're done, I claim that the problem is defective and doesn't belong in this section. To see this, compare your answer with Theorem 12.3 and conclude that one of a)-d) is violated, so this can't be a Sturm-Liouville problem. Then compare with the definition (3)-(5) on page 673 and verify that this problem doesn't match the definition.[1]

c) *12.5.8*

d) *12.6.15* Save yourself time and just write out the first 4 non-zero terms.

The Fourier Transform

a) *15.3.2*

b) *15.3.17* (*Hint:* Inverse transform.)

c) From class we know how to compute the Fourier Transform of a derivative in terms of the original Fourier Transform (see also the bottom of page 751). Let's now figure out what the derivative of the Fourier Transform is. Let $\mathcal{F}$ denote the complex Fourier Transform and suppose $f(x)$ has Fourier Transform $\mathcal{F}\{f(x)\}(k) = F(k)$. Show that $F'(k) = i\mathcal{F}\{xf(x)\}(k)$. That is, up to a multiple of i, multiplication by x and then Fourier Transforming is the same as differentiating. *Hint:* Differentiate under the integral sign.

d) Use ideas from c) to compute the (complex) Fourier Transform of the differential equation $y''(x) + x^2y(x) = 0$. In particular, show that this equation is the same in the transformed variable k as it is in x.

A quick word about L^2 In lecture I promised that L^2 was a special space, in particular it's the place where Fourier Theory works. These are a few quick problems to give some sense of why it's more interesting than other similar function spaces. Recall that in general $L^p(a, b)$ is defined to be the collection of all function $f(x)$ on (a, b) such that $\int_a^b |f(x)|^p \, dx$

[1] Of course, the answer you get is a complete orthogonal set; namely the Fourier basis. However, it isn't a complete orthogonal set by virtue of Sturm-Liouville theory. You need different techniques to show this.

exists as a finite number.

a) Consider $L^1(0,1)$. Show that $f(x) = \frac{1}{\sqrt{x}}$ is in this space. Now show that it's square is not. In particular, the "inner product" $< f, g >= \int_0^1 f(x)g(x)\,dx$ doesn't make sense here. Indeed, one can show that L^2 is the only L^p space where this inner product works.

Of course, the inner product $< f, g >= \int_a^b f(x)g(x)\,dx$ makes sense in other places than $L^2(a,b)$, for example it is defined on the continuous function $C([a,b])$ since the product of two continuous function is integrable. So there has to be another reason why L^2 is noteworthy:

b) For each positive integer k put $f_k(x) = \frac{1}{2} + \frac{1}{\pi}\tan^{-1}(kx)$. Sketch the graph of $f_k(x)$ for $k = 1, 2, 3$. Now let $k \to \infty$ and describe the resulting limiting function. In particular, note that limit is **not** continuous. Hence, even when we take well behaved limits in $C([a,b])$ we leave the space. It's a difficult result[2] that $L^2(a,b)$ however is well behaved under taking limits. In particular, forming things like infinite series works better in L^2 which allows things like Fourier Series to work.

[2]The "Riesz-Fisher Theorem", but whatever...