241 Homework 1 - Solutions and Discussion

Problem 12.2.2 *Solve the Sturm Liouville problem defined by $y'' + \lambda y = 0$ subject to the boundary conditions $y(0) + y'(0) = 0$ and $y(1) = 0$.*
Solution: For $\lambda < 0$ the equation has solutions in real exponentials or, equivalently, in cosh and sinh. For simplicity put $\delta^2 = -\lambda > 0$. The general solution is thus seen to be $y(x) = a\sinh(\delta x) + b\cosh(\delta x)$ from which we easily compute $y'(x) = \delta(a\cosh(\delta x) + b\sinh(\delta x))$. The condition $y(0) + y'(0) = 0$ then becomes $y(0) + y'(0) = b + \delta a = 0$, as $\sinh 0 = 0$ and $\cosh 0 = 1$. That is, $b = -\delta a$. Using this relation and plugging in for the second condition $y(1) = 0$ yields $y(1) = a(\sinh(\delta) - \delta\cosh(\delta)) = 0$. If $a = 0$ then $b = -\delta a$ forces b to be zero, so the solution is the trivial $y \equiv 0$ case. So assuming $a \neq 0$ we have $\sinh(\delta) - \delta\cosh(\delta) = 0$, or equivalently $\delta = \tanh(\delta)$. I now claim that this equation has no solutions for $\delta \neq 0$). Many people demonstrated this with a picture, which was fine, but let me show you that it can be done rigorously: Put $f(x) = x - \tanh(x)$. Observe that $f(0) = 0$. The derivative is seen to be $f'(x) = 1 - \frac{(e^2 - e^{-x})^2}{(e^2 + e^{-x})^2}$. Since e^x and e^{-x} are both greater than zero, the numerator of the second term is easily seen to be larger than the denominator, so the term is less than 1 and clearly greater than 0. Thus, $f'(x) > 0$ and so $f(x)$ is a strictly increasing function of x. Since $f(0) = 0$, $f(x)$ can have no zeros for $x > 0$, which is easily checked to be equivalent to $\delta = \tanh(\delta)$ having no non-trivial solutions.

For $\lambda = 0$ we have the general solution $y(x) = cx + d$. The boundary conditions are easily seen to force $c = -d$ and indeed $y(x) = x - 1$ is an eigenfunction.

For $\lambda > 0$ set $\lambda = \epsilon^2$ for simplicity. The general solutions is thus $y(x) = e\cos(\epsilon x) + f\sin(\epsilon x)$, so that $y'(x) = \epsilon(f\cos(\epsilon x) - e\sin(\epsilon x))$. The boundary conditions thus force $e + f\epsilon = 0$ and $e\cos\epsilon + f\sin\epsilon = 0$. Ignoring the trivial solution forces $\tan\epsilon = \epsilon$. A plot of $\tan x$ vs. x reveals an infinite number of solutions for $x > 0$, and these solutions correspond to (the positive square root of...) the eigenvalues of our problem. Call these solutions λ_n. Then our eigenfunctions are checked to be $\sqrt{\lambda_n}\cos\sqrt{\lambda_n}x - \sin\sqrt{\lambda_n}x$. Numerically, one finds the first four eigenfunctions to be approximately 20.19, 59.68, 118.90, and 197.86.

Problem 12.2.10 *State an orthogonality relation for the first two eigenvalues from the above problem.* The problem is already in self-adjoint form, so there is no weight function. Hence, the general orthogonality relation is just that the product of two distinct eigenfunctions integrated over $(0, 1)$ is zero. That is, the general orthogonality relation is

$$\int_0^1 \left(\sqrt{\lambda_m}\cos\sqrt{\lambda_m}x - \sin\sqrt{\lambda_m}x\right)\left(\sqrt{\lambda_n}\cos\sqrt{\lambda_n}x - \sin\sqrt{\lambda_n}x\right) dx = 0 \qquad (1)$$

Taking the square root of the first eigenvalue 20.19 to be approximately 4.493 and the square root of 59.68 to be approximately 7.725 and plugging these in for λ_1 and λ_2 in the above equation, maple gives back the number -.00002565, which is reasonably close to zero.

Problem 12.5.4: *Solve $y'' + \lambda y = 0$ subject to the boundary conditions $y(-L) = y(L)$ and $y'(-L) = y'(L)$. In particular, show that your set of orthogonal solutions is the Fourier Basis.*

Solution: For $\lambda < 0$ we have, just as in the above problem, solutions in cosh's and sinh's. Say $y(x) = c_1 \cosh \sqrt{-\lambda} + c_2 \sinh \sqrt{-\lambda}$. Using that sinh is odd and cosh is even we compute that the condition $y(-L) = y(L)$ tells us

$$c_1 \cosh \sqrt{-\lambda}L - c_2 \sinh \sqrt{-\lambda}L = c_1 \cosh \sqrt{-\lambda}L + c_2 \sinh \sqrt{-\lambda}L \tag{2}$$

so after canceling we get $2c_2 \sinh \sqrt{-\lambda}L = 0$. The condition that $y'(-L) = y'(L)$ is seen, by differentiating the general solution, to give

$$-c_1\sqrt{-\lambda} \sinh \sqrt{-\lambda}L + c_2\sqrt{-\lambda} \cosh \sqrt{-\lambda}L = c_1\sqrt{-\lambda} \sinh \sqrt{-\lambda}L + c_2\sqrt{-\lambda} \cosh \sqrt{-\lambda}L \tag{3}$$

so after canceling we get $2c_1\sqrt{-\lambda} \sinh \sqrt{-\lambda}L = 0$. For non-trivial solutions at least one of c_1 or c_2 must be non-zero. Either of these cases forces, by the above observations, that $\sinh \sqrt{-\lambda}L = 0$. But by looking at the graph of sinh, one sees that $\sinh x = 0$ only when $x = 0$, so this condition solution is incompatible with the condition $\lambda < 0$ and there are no non-trivial solutions.

For the $\lambda = 0$ case our differential equation becomes $y'' = 0$ and so $y(x) = mx + b$. The $y(L) = y(-L)$ condition forces $mL + b = -mL + b$, so $2mL = 0$, which forces $m = 0$. It's easy to check that the constant solution $y(x) = c$ is indeed a consistent solution, and indeed this only one for this case.

For the $\lambda > 0$ case our solutions to the general equation come in the form of sin's and cos's. Say $y(x) = c_1 \cos \sqrt{\lambda}x + c_2 \sin \sqrt{\lambda}x$. The condition $y(-L) = y(L)$ yields (using even-ness and odd-ness):

$$c_1 \cos \sqrt{\lambda}L - c_2 \sin \sqrt{\lambda}L = c_1 \cos \sqrt{\lambda}L + c_2 \sin \sqrt{\lambda}L \tag{4}$$

so after canceling we get $2c_2 \sin \sqrt{\lambda}L = 0$. The condition $y'(-L) = y'(L)$ is seen, by differentiating the general solution, to give

$$\sqrt{\lambda}c_1 \sin \sqrt{\lambda}L + c_2\sqrt{\lambda} \cos \sqrt{\lambda}L = -\sqrt{\lambda}c_1 \sin \sqrt{\lambda}L + \sqrt{\lambda}c_2 \cos \sqrt{\lambda}L \tag{5}$$

so after canceling we get $2c_1\sqrt{\lambda}\sin\sqrt{\lambda}L = 0$. For non-trivial solutions, we cannot have both $c_1 = 0$ and $c_2 = 0$, and if *either* of these is non-zero it forces the condition $\sin\sqrt{\lambda}L = 0$, which means that $\lambda = (\frac{n\pi}{L})^2$. This problem is different from the previous one (and indeed, from most problems of this type) in that we cannot eliminate either c_1 or c_2 - they both can be non-zero and still give a valid solution. Recall that we're seeking out solutions which describe, in essence, a complete orthogonal set (i.e. a Fourier Basis), so it suffices to exhibit a *linearly independent* set of solutions. The easiest way to do this is to take $c_1 = 0$, $c_2 = 1$ and also $c_1 = 1$, $c_2 = 0$. These cases give eigenfunctions, respectively, $\sin\frac{n\pi x}{L}$ and $\cos\frac{n\pi x}{L}$.

Combining solutions for the $\lambda = 0$ and $\lambda > 0$ cases gives us the complete orthogonal set $\left\{1, \sin\frac{1\pi x}{L}, \sin\frac{2\pi x}{L}, \sin\frac{3\pi x}{L}, \ldots, \cos\frac{1\pi x}{L}, \cos\frac{2\pi x}{L}, \cos\frac{3\pi x}{L}, ..\right\}$ on $[-L, L]$.

Note that for each non-zero eigenvalue there are two eigenfunctions (the sin and the cos), contradicting that Sturm-Liouville problems have non-degenerate eigenvalues. The issue is that the given boundary conditions here are easily checked to not conform to the regular boundary conditions of a Sturm-Liouville problem.

Problem 12.5.8: *Find the eigenvalues and eigenfunctions of* $y'' + y' + \lambda y = 0$ *subject to* $y(0) = 0$ *and* $y(2) = 0$. *Then put the equation in self-adjoint form and exhibit a complete set of orthogonal solutions.*

Solution: The roots of the characteristic equation of the given differential equation are seen to be $\frac{-1\pm\sqrt{1-4\lambda}}{2}$. Hence, we have different behavior for $\lambda < \frac{1}{4}$, $\lambda = \frac{1}{4}$, and $\lambda > \frac{1}{4}$ due to the possibility of non-real roots. Also note that $\lambda = 0$ must be dealt with separately due to the disappearance of the y term.

For $\lambda = \frac{1}{4}$ we have the general solution $y(x) = ae^{-\frac{x}{2}} + xbe^{-\frac{x}{2}}$. The boundary conditions force $a = 0$ and $ae^{-1} + 2be^{-1} = 0$, which is easily checked to have only the trivial solution, in which case $y \equiv 0$.

For $\lambda < \frac{1}{4}$ and non-zero, we have the general solution $y(x) = ce^{\frac{-1+\sqrt{1-4\lambda}}{2}x} + de^{\frac{-1-\sqrt{1-4\lambda}}{2}}x$, from which the boundary conditions are easily checked to force $c = d = 0$ similar to the above case since all factors in the exponents are real and e^x is a strictly increasing function of x and thus $e^x = e^y$ if and only if $x = y$..

For $\lambda > \frac{1}{4}$ The general solution is, by Euler's Formula, $y(x) = fe^{-\frac{x}{2}}\cos\sqrt{4\lambda - 1}x + ge^{-\frac{x}{2}}\sin\sqrt{4\lambda - 1}x$. $y(0) = 0$ forces $f = 0$. The other condition forces $y(2) = \frac{g}{e}\sin 2\sqrt{4\lambda - 1} =$, which implies that $2\sqrt{4\lambda - 1} = n\pi$ for $n = 1, 2, 3, ...$ Hence, the eigenvalues are seen to be

$\frac{(n\pi)^2}{16} + \frac{1}{4}$ with corresponding eigenfunctions $e^{-\frac{x}{2}} \sin \frac{n\pi x}{2}$.

The self-adjoint form of the given ODE can be found either by using the standard substitution/trick, but it's not hard to see by inspection. Specifically, it's just $(e^x y')' + \lambda e^x y = 0$.

Our weight function here is $\rho(x) = e^x$ from above, so the orthogonality relation is

$$\int_0^2 (e^{-\frac{x}{2}} \sin \frac{m\pi x}{2})(e^{-\frac{x}{2}} \sin \frac{n\pi x}{2}) e^x \, dx = 0 \tag{6}$$

for $m \neq n$. The exponentials cancel, and we're left with

$$\int_0^2 \sin \frac{m\pi x}{2} \sin \frac{n\pi x}{2} \, dx \tag{7}$$

for $m \neq n$.

Problem 12.6.15: *Expand f(x)=x in a Legendre Series.*

Solution: The first few Legendre polynomials can be looked up. The morally correct way to do this problem would be to use Fourier's trick from scratch, but since this has already been worked out in the section for Legendre polynomials, let's just make use of their formula. Plugging in, we get:

$$c_0 = \frac{1}{2} \int_0^1 x P_0(x) \, dx = \frac{1}{2} \int_0^1 x \, dx = \frac{1}{4} \tag{8}$$

$$c_1 = \frac{3}{2} \int_0^1 x P_1(x) \, dx = \frac{3}{2} \int_0^1 x^2 \, dx = \frac{1}{2} \tag{9}$$

$$c_2 = \frac{5}{2} \int_0^1 x P_2(x) \, dx = \frac{5}{2} \int_0^1 \frac{1}{2}(3x^3 - x) \, dx = \frac{5}{16} \tag{10}$$

$$c_3 = \frac{7}{2} \int_0^1 x P_3(x) \, dx = \frac{7}{2} \int_0^1 \frac{1}{2}(5x^4 - 3x^2) \, dx = 0 \tag{11}$$

$$c_4 = \frac{9}{2} \int_0^1 x P_4(x) \, dx = \frac{9}{2} \int_0^1 \frac{1}{8}(35x^5 - 30x^3 + 3x) \, dx = -\frac{3}{32} \tag{12}$$

And thus $f(x) = x \approx \frac{1}{4} P_0(x) + \frac{1}{2} P_1(x) + \frac{5}{16} P_2(x) - \frac{3}{32} P_4(x)$

Problem 15.3.2 *Compute the Fourier Integral Representation of $f(x)$ which is equal to 4 on $(\pi, 2\pi)$ and zero elsewhere.*

Solution: We first compute the Fourier Amplitudes

$$A(\alpha) = \int_{\pi}^{2\pi} 4\cos\alpha x\, dx = 4\frac{\sin 2\pi\alpha - \sin\pi\alpha}{\alpha} \tag{13}$$

$$B(\alpha) = \int_{\pi}^{2\pi} 4\cos\alpha x\, dx = 4\frac{-\cos 2\pi\alpha + \cos\pi\alpha}{\alpha} \tag{14}$$

So the Fourier Integral Representation arises from inverse transforming these expressions:

$$f(x) = \frac{4}{\pi}\int_0^{\infty} \frac{(\sin 2\pi\alpha - \sin\pi\alpha)\cos\alpha x + (-\cos 2\pi\alpha + \cos\pi\alpha)\sin\alpha x}{\alpha}\, d\alpha \tag{15}$$

which simplifies somewhat via trig identities to

$$f(x) = \int_0^{\infty} \frac{\sin\alpha(2\pi - x) - \sin\alpha(\pi - x)}{\alpha}\, d\alpha \tag{16}$$

Problem 15.3.17 *Solve the integral equation $\int_0^{\infty} f(x)\sin\alpha x\, dx = e^{-\alpha}$. That is, find $f(x)$ such that the above equation holds.*

Solution The expression on the left hand side of the equation above is exactly the Fourier Sine Transform of $f(x)$. Hence, we apply the inverse sine transform to both sides to obtain

$$f(x) = 2\pi\int_0^{\infty} e^{\alpha}\cos\alpha x\, d\alpha \tag{17}$$

The expression on the right can be computed exactly using integration by parts twice. It is found to be $\frac{2}{\pi}\frac{1}{1+x^2}$.

Compute the Derivative of the Fourier Transform **Solution:** $\frac{\partial}{\partial k}\int_{-\infty}^{\infty} e^{ikx} f(x)\,dx = \int_{-\infty}^{\infty} \frac{\partial}{\partial k}(e^{ikx} f(x)\,dx) = i\int_{-\infty}^{\infty} e^{ikx} x f(x)\,dx = i\mathcal{F}(xf(x))(k)$.

Fourier Transform the differential equation $y''(x) + x^2 y(x) = 0$.

Solution: By Linearity $\mathcal{F}(y''(x) + x^2 y(x)) = \mathcal{F}(y''(x)) + \mathcal{F}(x^2 y(x))$. From the book, the Fourier transform of $y''(x)$ is $-k^2 Y(k)$. By Differentiating the Fourier transform of $y(x)$ *twice* in the same fashion as the above problem, it is easy to see that $Y''(k)$ is just minus the Fourier transform of $x^2 y(x)$. That is, $\mathcal{F}(x^2 y(x))(k) = -Y''(k)$. Hence the Fourier transform of the differential equation is $-k^2 Y(k) - Y''(k) = 0$. That is, it's the same equation only in Fourier space. Weird, huh?

There's no question about the geometry of L^2 on the exam, so I'll omit solutions to those problems. If you still have questions about them, please ask.