The Transverse Invariant and Bindings of Open Books

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- Talk about some generalizations and open questions related to these invariants

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- Can prove nice vanishing and nonvanishing theorems
- These invariants play well with other known invariants

Contact Structures

Definition

Let Y be an oriented 3-manifold. A (cooriented) plane field ξ on Y is called a **contact structure** if

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Example

(S^3, ξ_{std}), $\quad \xi_{\text{std}} = \{$complex tangencies to $S^3 \subset \mathbb{C}^2$ $\}$.

Legendrian and Transverse Knots

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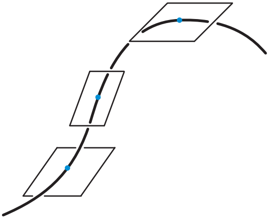

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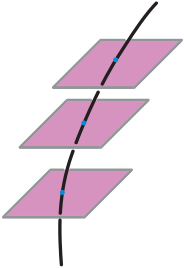

- A knot L is **Legendrian** if, for each point p in L,

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- A knot B is **transverse** if, for each point p in B,

$$T_pB \pitchfork \xi_p.$$

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We have powerful non-classical invariants for Legendrian knots (e.g. Chekanov-Eliashberg DGA), but transverse invariants are harder to come by.

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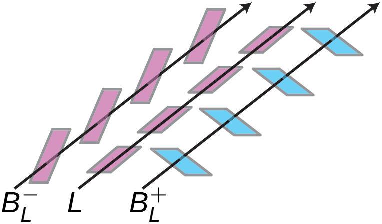

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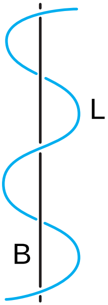

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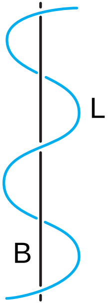

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Two Legendrian approximations related by “negative stabilization”.

Open Books

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Remark: Alternatively, an open book can be specified by naming a fiber surface S and a monodromy map $\phi : S \to S$.

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- Tangents to the fibers $S_\theta = \pi^{-1}(\theta)$ appropriately approximate ξ

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Theorem (Giroux)

For a fixed 3-manifold Y, there exists a 1-1 correspondence between

$$\left\{ \begin{array}{l} \textit{open book decompositions of} \\ Y \textit{ up to "positive stabilization"} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \textit{Contact Structures on} \\ Y \textit{ up to isotopy} \end{array} \right\}$$

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- Get **transverse invariants** $\mathfrak{T}(B)$ and $\widehat{\mathfrak{T}}(B)$!

A Computation

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Theorem (V)

Let (B, π) be an open book decomposition with connected binding for a contact manifold (Y, ξ). Then the transverse invariant $\widehat{\mathfrak{T}}(B)$ is nonvanishing.

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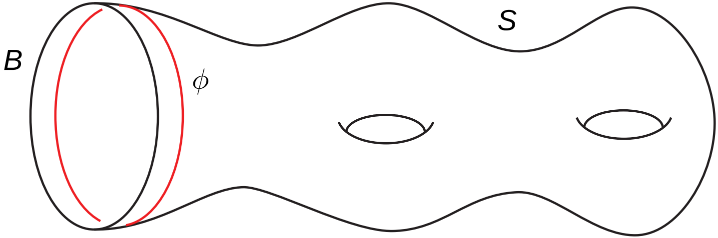

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- A contact manifold has **positive Giroux torsion** if there exists an embedding

$$\phi : (T^2 \times [0,1], \xi_{2\pi}) \to (Y, \xi),$$

where $\xi_{2\pi} = \ker(\cos(2\pi t)\,dx + \sin(2\pi t)\,dy)$

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Let (B, π) be an open book decomposition with connected binding for (Y, ξ). Then the complement $(Y - B, \xi|_{Y-B})$ has no Giroux torsion.

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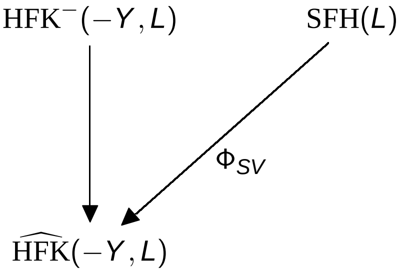

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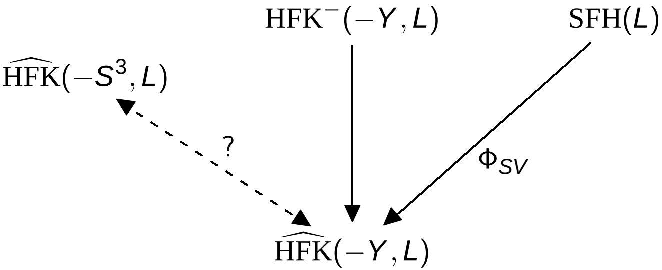

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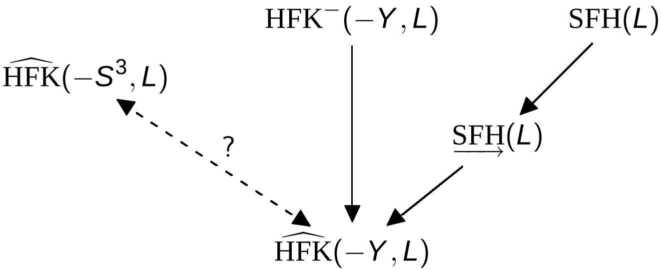

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