3-Manifolds Pizza Talk

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Let's get started with a definition.

Definition 1.1. We say a 3-Manifold M^3 is **prime** if $M \cong M_1 \sharp M_2$ implies M_1 or M_2 is S^3.

So why do we care about prime manifolds at all? What makes them interesting enough to give an entire pizza talk about? Well, here's a spoonful of motivation!

Theorem 1.1 (Prime Decomposition Theorem)**.** *[Kneser Milnor] Every 3-manifold M is a finite connected sum of prime 3-manifolds.*

$$M = M_1 \sharp \ldots \sharp M_n$$

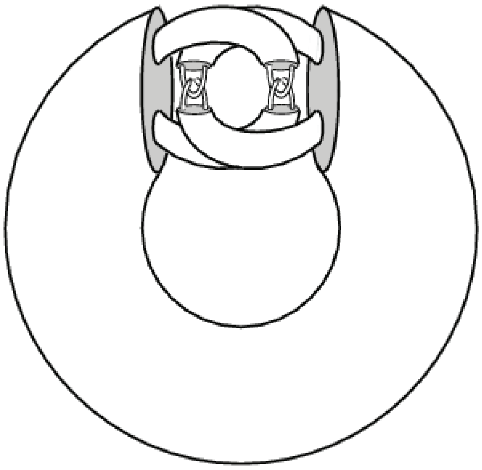
Furthermore, any two decompositions are unique up to order (and summing on S^3's).

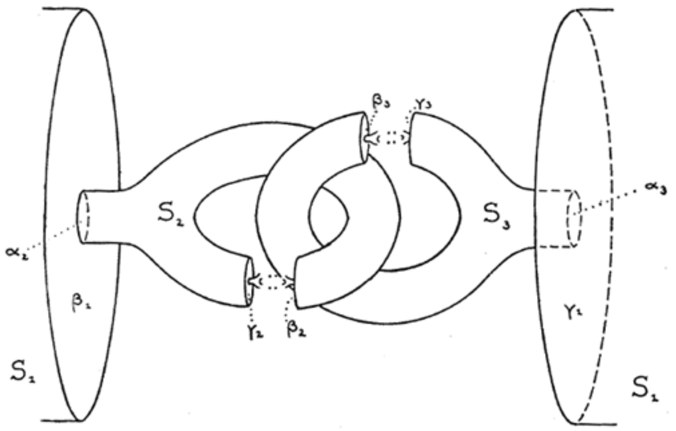
This is obviously a REALLY amazing theorem. It basically tells us that if our goal is to understand 3-manifolds, then we can restrict our attention to prime ones.

The purpose of today's talk will be to investigate these prime manifolds and learn some of their topological properties along the way. To get this discussion started, let's recall a classical theorem reminiscent of the "Jordan curve theorem" for S^2.

Theorem 1.2 (Schönflies Theorem)**.** *If $S^2 \subset S^3$ is a smooth 2-sphere, then the closure of each component of $S^3 \backslash S^2$ is diffeomorphic to a 3-ball.*

Remark 1.1. This is false if we don't assume smooth. Take the Alexander horned-sphere for example.





Corollary 1.3. *S^3 and $\mathbb{R}^3$ are prime.*

Definition 1.2. We say M is **irreducible** if every embedded 2-sphere in M bounds a 3-ball.

Remark 1.2. Clearly if M is irreducible, then M is prime. What about the converse?

Theorem 1.4. *M is prime $\Leftrightarrow$ M is irreducible or $M \cong S^1 \times S^2$.*

Proof. Proving ($\Rightarrow$) would take us far a field.

For ($\Leftarrow$), we already observed that if M is irreducible, then M is prime. So all that we really need to show is that $S^1 \times S^2$ is prime.

Suppose $S^1 \times S^2 \cong M_1 \sharp M_2$. Then we know by Van-Kampen's Theorem that $\mathbb{Z} = \pi_1(S^1 \times S^2) = \pi_1(M_1) * \pi_1(M_2)$. But the only way the free-product of two groups can be $\mathbb{Z}$ is if one of the groups is trivial (cardinality). So let's suppose it's M_1.

Now let's look at the universal $\mathbb{R} \times S^2$ of $S^1 \times S^2$. We can think of $\mathbb{R} \times S^2$ as $\mathbb{R}^3 - \{0\}$. Then since $\pi_1(M_1') = 1$, we have that M_1' can be lifted to $\mathbb{R} \times S^2 = \mathbb{R}^3 - \{0\} \subset \mathbb{R}^3$. But $\mathbb{R}^3$ is irreducible, so $M_1' = B^3$. □

This last result is really great. It tells us that as far as prime manifolds go, we've got a lot of control on their topology.

Here's a theorem that we can use to construct new prime manifolds from ones we already know.

Theorem 1.5. *If $\widetilde{M} \to M$ is a regular covering space, then $\widetilde{M}$ is irreducible if and only if M is irreducible.*

Proof. ($\Rightarrow$) Hard!

($\Leftarrow$) Here's a sketch. We lift our S^2 up to $\widetilde{M}$. Now, in $\widetilde{M}$, we have many copies of our original S^2 (one for each element of the Deck group. Furthermore, since $\widetilde{M}$ is irreducible, they all bound 3-balls.

Exercise: Show that these 3 balls can be chosen so that they are all mutually disjoint, and translates of one another under the Deck group.

Once we have that, then it's clear that we can project one of these balls back down to M. Then $S^2 \subset M$ bounds this ball. □

Corollary 1.6. *Lens spaces are irreducible*

Now for another big-daddy theorem!

Theorem 1.7 (Sphere Theorem)**.** *[Papakyriakopolous '57, Whitehead '58] Let M be a 3-manifold with $\pi_2(M) \neq 0$, then there exists an embedded $e : S^2 \hookrightarrow M$ such that $[e] \neq 0$ in $\pi_2(M)$*

Corollary 1.8. *If M is prime, then either $\pi_2(M) = 0$, or $M \cong S^1 \times S^2$.*

Proof. Suppose M is prime and not $S^1 \times S^2$. Now suppose there exists an $f : S^2 \to M$ with $[f] \neq 0 \in \pi_2(M)$. By the sphere theorem, we then have that there exists an embedded $e : S^2 \hookrightarrow M$ with $[e] \neq 0 \in \pi_2(M)$. But M is prime and not $S^1 \times S^2$, so M is irreducible. Thus, our S^2 bounds a 3-ball in M, contradicting the assumption that $[e] \neq 0 \in \pi_2(M)$, and therefore $[f] \neq 0 \in \pi_2(M)$. □

Corollary 1.9. *Let M be a closed, connected, orientable, irreducible 3-Manifold with universal cover $\widetilde{M}$.*
(i) If $\pi_1(M)$ is infinite, then M is a $K(\pi, 1)$, ($\pi_i(M) = 0$ for all $i \geq 2$) (true even if M has ∂).
(ii) If $\pi_1(M)$ is finite, then $\widetilde{M}$ is a homotopy 3-sphere (and therefore S^3 if you believe the Poincaré conjecture).

Proof. (i) Since M is irreducible, the Sphere Theorem tells us that $\pi_2(M) = 0$. We also have $\pi_3(M) = \pi_3(\widetilde{M}) = H_3(\widetilde{M})$ (the last equality being true by Hurewicz). Now, since $\pi_1(M)$ is infinite, we have that $\widetilde{M}$ is non-compact. Thus, $H_3(\widetilde{M}) = 0$, and we see that $\pi_3(M) = 0$. Continuing, we see that $\pi_4(M) = \pi_4(\widetilde{M}) = H_4(\widetilde{M}) = 0$ (since $\widetilde{M}$ is a 3-manifold). Repeating this process forever, we see that $\pi_i(M) = 0$ for all $i \geq 2$. Thus, M is a $K(\pi, 1)$.

(ii) If $\pi_1(M)$ is finite, then $\widetilde{M} \to M$ is a finite-sheeted covering of M. In particular, we have that since M is closed, $\widetilde{M}$ is closed. Thus, since $\widetilde{M}$ is a closed, simply connected 3-manifold, M is a homotopy 3-sphere. □

(For this last fact, we have that $H_1(M) = H_2(M) = 0$, and that $\pi_3(M) = H_3(M) = \mathbb{Z}$ and that the isomorphism is given by the Hurewicz map. Namely, we get a degree 1 map $S^3 \to M$ inducing an isomorphism on all homology groups. Thus, since both manifolds are simply connected, Whitehead's theorem tells us that they are homotopy equivalent.)

Moral of the story: If M is a 3-manifold, then M can be broken down into a sum of prime manifolds, each of which is either $S^1 \times S^2$ or a $K(\pi, 1)$. Furthermore, using stuff I'm not talking about here, we can put strong restrictions of the possible π_1's that these manifolds can have. And yet the classification remains so very unsolved.