Swimming the Salmon River

It's been a hot day on the Salmon River. See Figure 1 (figures on last page). Your rafting party has stopped for lunch on a pleasant stretch of sand, and you are basking in the warmth of the sun on the beach. After a few minutes, you notice two members of the opposite sex pull their kayaks out onto the beach directly across from you. Suddenly, the beach on the other side of the river seems more inviting than the one on your side. Your rafting companions don't share that interest. You ask yourself, "How fast must I swim to get across and not be swept downstream into Fiddle Creek Rapids?" It would completely ruin the panache you show by swimming the river if you died in the attempt.

Let's derive a system of differential equations that describes your velocity at any point in the river. The first step is to introduce a coordinate system. Let the river flow in the positive y-direction or northward, locate the kayakers at the point (0, 0) on the west beach, and locate your position on the east beach at (w, 0). See Figure 2(a). Suppose that the river is *w* feet across and is flowing at a constant rate of v_r ft/s. Suppose further that you swim at a rate of v_s ft/s relative to the river and that your *velocity is always directed toward the kayak party.* See Figure 2(b). We want to know the answer to the question "What does v_s have to be in order for there to be a solution from (w, 0) to (0, 0)?"

If you are at the point *(x(t), y(t))* at time *t*, then your velocity vector **v** at this point is the sum of two vectors $\mathbf{v} = \mathbf{v}_s + \mathbf{v}_r$, representing, in turn, your speed in the direction of your swimming, $\mathbf{v}_s$, and in the direction of the river ($\mathbf{v}_r$). Because your swimming direction is always directed toward the kayaks at (0, 0), $\mathbf{v}_s$ has components in both the x- and y-directions, while the river's velocity $\mathbf{v}_r$. has only a component in the y-direction. Using the fact that *dx/dt* is the component of **v** in the x-direction and *dy/dt* is the component of **v** in the y-direction we see from Figure 2(b) that:

$$\mathbf{v} = \left(\frac{dx}{dt}, \frac{dy}{dt}\right) = \mathbf{v}_s + \mathbf{v}_r = (-v_s\cos\theta, -v_s\sin\theta) + (0, v_r) = (-v_s\cos\theta, v_r - v_s\sin\theta) \qquad (1)$$

where the magnitudes $|\mathbf{v}_r| = v_r$ and $|\mathbf{v}_s| = v_s$ are speeds. By using the fact that the corresponding components in (1) are equal and that $\cos\theta = \frac{x}{\sqrt{x^2+y^2}}$, $\sin\theta = \frac{y}{\sqrt{x^2+y^2}}$ we've constructed the system of first-order differential equations

$$\frac{dx}{dt} = -v_s\frac{x}{\sqrt{x^2+y^2}} \qquad \frac{dx}{dt} = v_r - v_s\frac{y}{\sqrt{x^2+y^2}} \qquad (2)$$

Because $\frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt}$ the solution curves of this system of differential equations satisfy the single first-order equation

$$\frac{dy}{dx} = \frac{y - \left(v_r/v_s\right)\sqrt{x^2+y^2}}{x} \qquad (3)$$

Your goal is to now solve equation (3) and determine what values of v_r/v_s allow you to reach the kayakers and avoid the embarrassment of being dashed on the rocks below.

PROBLEM 1. Because w is the initial distance between you and the kayakers, the initial condition associated with (3) is y(w) = 0. Solve this initial-value problem. (Maple's **dsolve** command is a real help here!)

PROBLEM 2 (Maple—**DEplot**). Without loss of generality we can let w = 1 so that your position on the east beach is (1, 0). Use Maple to graph the solution in Problem 1 in the case v_r = 1 for various values of v_s-that is, for $0 < v_s < 1$ $v_s = 1$, and $v_s > 1$. Determine under what conditions you either make it to the kayaks or don't. Repeat this problem for other values of v_r. Make a conjecture on the relationship between v_r and v_s necessary to ensure that you safely make it to the kayaks.

PROBLEM 3. Prove that if you swim at a speed v_s greater that the current speed of the river-that is, $v_s > v_r$ then you will reach the kayaks. Explain from a physical perspective why this condition is necessary even though you are not swimming directly into the current.

Your friend Bubba decides to follow you and swim to the west beach to meet the kayakers. Bubba does not understand why you swam directly at the kayakers. He thinks that he must be able to reach the opposite beach by just swimming directly west (relative to the river) at constant rate v_s. He is confident that he can swim fast enough to avoid being swept into Fiddle Creek Rapids 3 miles downstream from the point (1, 0). He plans to simply walk to the kayakers' position when he hits the beach.

PROBLEM 4. Show that the model for the path Bubba takes in the river is

$$\frac{dy}{dx} = -\frac{v_r}{v_s}$$

PROBLEM 5. The current speed v_r of a river is usually not a constant. Rather, an approximation to the current speed (measured in miles per hour) could be a function such as $v_r = 30x(1-x), 0 \le x \le 1$, the values of which are smallest at the shores (in this case, x = 0 and x = 1) and largest in the middle of the river. Assume that Bubba starts from (1, 0) and that v_s = 2 mph. Solve the DE in Problem 4 with v_r as given above. Will he make it across the river, or will he be swept into the rapids? If he makes it across the river, how far will he have to walk to reach the kayakers?

Figure 1: Rafting on the Salmon

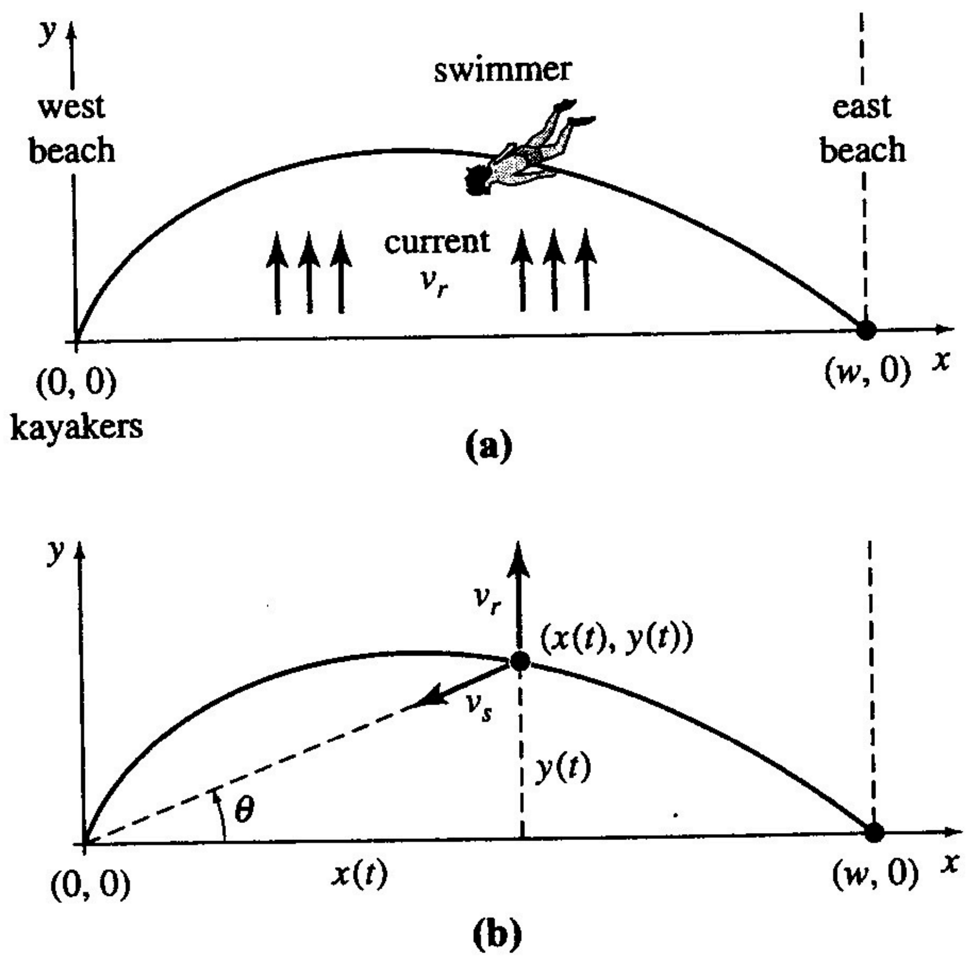

Figure 2: Path of Swimmer

Adapted from Zill, D and Cullen, M, "Differential Equations, 6th Ed."