

You are a fisheries manager for the state of Alaska, and it is your job to help establish regulations for commercial salmon fishing in territorial waters near Glacier Bay, Alaska. Your problem is that you never know how many fish are out there or which species will dominate in a given year. People who fish salmon for a living depend on the limits you set to put food on their tables and pay their bills. On the other hand, you have a fishery that could collapse if you set limits that are too high.

Commercial fishing exploits a natural and wild resource. In contrast to, say, a cattle farmer who can breed new calves to replenish the herd continually, fishermen taking a wild resource cannot control their supply except by crude harvest limitations. Thus commercial and recreational fishing are regulated in an attempt to limit the harvesting and preserve the resource. The effects of harvesting a renewable natural resource such as fish can be modeled using a modification of the logistic equation:

$$\frac{dP}{dt} = rP \left(1 - \frac{P}{K} \right) - H(P) \quad (1)$$

where: $P(t)$ = population of salmon (in thousands) at time t (in years); K = carrying capacity (max sustainable population; in thousands); r = growth rate; and $H(P)$ = harvesting rate.

But first, consider what happens to the population without harvesting (i.e., let $H(P) = 0$). Let

$$l(P) = rP \left(1 - \frac{P}{K} \right) \quad (2)$$

and

PROBLEM 1: Graph $y = l(p)$ and use this graph to sketch the solution of equation (1) with $H(P) = 0$ for several initial values; choose suitable values for r and K ($r = 1$ and $K = 1$ work to get the *shape* of the solution curves and this curve is easily scaled to produce solutions for different choices for r and K). What are the *critical points* for this equation (i.e., for what value(s) of P is $dP/dt = 0$) and what is the significance of these values? Are these critical points *stable* or *unstable*? That is, does a small change in the value of P away from the critical point result in a solution that returns to the critical value eventually (*stable*) or a solution which moves *away from* the critical point and, possibly, moves toward a stable critical point (*unstable*)?

PROBLEM 2: Interpret the solution curves sketched in Problem 1. Start with the equilibrium solutions (i.e., $dP/dt = 0$; no change in P) produced by certain choices of P in Problem 1, then discuss the solutions with non-equilibrium initial conditions.

Now that you understand what happens without human intervention, you begin to consider what happens if we start harvesting the salmon. You begin by assuming that $r = 1$ and that the carrying capacity is $K = 1000$. Your objective is to deter-

mine a harvest rate for the fishing industry. Thus instead of assigning a number to the harvest rate, you simply assume that the harvesting occurs at a constant rate of h thousand salmon per year. With $H(P) = h$, (1) becomes

$$\frac{dP}{dt} = rP \left(1 - \frac{P}{K} \right) - h \quad (3)$$

The DE in (3) is called the constant-harvest model. Your goal is to understand what happens to the salmon population as h increases.

PROBLEM 3 (Maple makes this easier). In (3), let $f_h(p) = rP \left(1 - \frac{P}{1000} \right) - h$. Examine $f_h(P)$ for $0 < h < 300$. Describe what happens to the critical points of (3). Relate this behavior to the population model. In particular, for each harvesting level h , what initial population levels ultimately lead to extinction, and how does this change with h ? After all, it is hard to make money fishing for salmon if there are none...

PROBLEM 4. For what value of h is there only one critical point? What is the behavior of the solutions of h slightly smaller than and slightly larger than this value? Again, interpret this in terms of the model for the salmon population.

One thing that you need to care about is a dependence on specific numbers in the model. You don't actually know the ocean's carrying capacity K or a species growth rate r . Studies and experiments suggest values for these parameters, but even then they are only estimates. It is very important for you to understand the behavior of the model as a whole. Otherwise, you could make a mistake in parameter value and wipe out the fishery.

PROBLEM 5. Repeat Problem 3 without assigning numerical values to r and K . Use calculus techniques to sketch a graph of $f_h(P)$. The constant-harvest model (3) that you have examined corresponds to a simple approach to licensing in which fishermen are allowed a constant take, regardless of the time required for that. Another approach is to assume that harvesting is proportional to the population present. In other words, instead of allowing the same number of fish to be harvested each year, you allow only a fraction of the present population to be caught. In this scenario we write $H(P) = \alpha P$, $0 \leq \alpha < 1$, and (1) becomes:

$$\frac{dP}{dt} = rP \left(1 - \frac{P}{K} \right) - \alpha P \quad (4)$$

The differential equation in (4) is called the proportional harvesting model.

PROBLEM 6. Compute the critical points of the proportional harvesting model (4) and classify their stability. Interpret your results in terms of the salmon population.

PROBLEM 7: Now it's time to prepare your report and suggest a policy for harvesting the salmon in your charge. Compare and contrast the two harvesting strategies. Be sure to make note of the strengths, weaknesses and limitations of each model. Brainstorm with your colleagues (i.e., fellow students) some other possible strategies and model theses as well. Present your conclusions as if your report were directed to the actual cabinet level official to whom a person in this position would report.