Lebesgue Measure

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1 Overview

Measure theory aims to quantify the size of sets of points. It generalizes the notion of length, area, and volume of classical geometric shapes to more general types of sets that appear in the analysis of functions. Besides squares and spheres and cylinders, we want to be able to measure Cantor sets and fractals and other sets that do not have smooth boundaries.

The most important measure for the purposes of analysis is *Lebesgue measure*, introduced by Henri Lebesgue in his doctoral thesis of 1902, as part of the foundation for his new technique of integration. In these notes we give a careful construction of the Lebesgue measure in Euclidean space $\mathbb{R}^n$. Most constructions of Lebesgue measure take as their point of departure the familiar notion of the volume of a rectangular box. From there, one gradually extends to more and more complicated sets.

In section 2, we study how to measure sets that can be divided into rectangular cells. This is one way to analyze the area inside a circle, for example: by dividing it up as an infinite collection of smaller and smaller squares. It turns out that all open sets can be measured in this way, but that not all closed sets are amenable to this approach.

Examples like the Cantor set show that closed sets may have no interior, and therefore no rectangle at all will fit inside. To deal with such sets, in section 3 we define outer and inner measures, which are defined for arbitrary sets. Now open and closed sets end up playing complementary roles.

We are now ready to define Lebesgue measure, which is done in section 4. Essentially, to measure a set we squeeze it between a compact set (from inside) and an open set (from outside). If the difference in measure between the compact and open set can be made arbitrarily small, the inner and outer measures of the set are the same, and the set is Lebesgue measurable. The key property of Lebesgue measure, proven in section 4, is that it is additive for countable disjoint unions of measurable sets.

Finally, in section 5 we give a characterization of measurable sets, and will see that virtually all sets that play a role in analysis are Lebesgue measurable.

2 Content of Cellular Sets

As usual $[a, b)$ denotes the half-open interval of numbers $x \in \mathbb{R}$ with $a \leq x < b$. We allow $a \leq b$. The *length* of a half-open interval $[a, b)$ is simply $b - a$). Because there are no numbers x with $a \leq x < a$, we have $[a, a) = \emptyset$. We see that the length of $\emptyset = [a, a)$ is zero.

A *cell* in $\mathbb{R}^n$ is a product of half-open intervals

$$S = [a_1, b_1) \times [a_2, b_2) \times \cdots \times [a_n, b_n),$$

where $a_i \leq b_i$. The intervals $[a_i, b_i)$ are called the *sides* of S. The *content* of S is the product of the length of its sides,

$$\nu(S) = (b_1 - a_1)(b_2 - a_2) \cdots (b_n - a_n).$$

Again, $\emptyset = [a_1, a_1) \times \cdots [a_n, a_n)$ is a cell with content zero. With this convention the intersection of any two cells is again a cell (possibly empty).

A *partition* of an interval $[a, b) \subset \mathbb{R}$ is a finite collection of disjoint intervals $[x_i, x_{i+1})$ whose union is $[a, b)$. Such a partition corresponds to a finite strictly increasing sequence,

$$a = x_0 < x_1 < \cdots x_{n-1} < x_n = b.$$

A partition of a cell S in $\mathbb{R}^n$ is obtained by partitioning each of the sides of S,

$$a_i = x_1^i < x_2^i < \cdots < x_{N_i}^i = b_i,$$

resulting in a collection P of $N = N_1 N_2 \cdots N_n$ disjoint cells of the form

$$[x_{j_1}^1, x_{j_1+1}^1) \times [x_{j_2}^2, x_{j_2+1}^2) \times \cdots \times [x_{j_n}^n, x_{j_n+1}^n).$$

If we refer to "a partition P of S", then P denotes the set of subcells of S in the partition, and we will often assume that those subcells are numbered, at random,

$$P = \{S_1, S_2, \ldots, S_N\}.$$

The construction of Lebesgue measure in Euclidean space $\mathbb{R}^n$ has as its starting point the idea that the measure of a cell S must be equal to its content $\nu(S)$. We will first prove that content is well-behaved for disjoint unions of cells.

Lemma 1 *Let the cell* $S = S_1 \cup S_2 \cup \cdots \cup S_N$ *be the disjoint union of finitely many subcells (not necessarily a partition). Then*

$$\nu(S) = \nu(S_1) + \nu(S_2) + \cdots + \nu(S_N).$$

Proof. If the union of cells S_i is a *partition* of S, then this follows immediately from the distributive property $a(b+c) = ab + ac$.

In the general case, for $i = 1, 2, \ldots, N$, let

$$S_i = [c_1^{(i)}, d_1^{(i)}) \times \cdots \times [c_n^{(i)}, d_n^{(i)}).$$

By putting the various boundaries of sides with the same coordinate

$$\{c_j^{(1)}, d_j^{(1)}, \ldots, c_j^{(N)}, d_j^{(N)}\}$$

in ascending order (removing duplicates), we obtain a partition of each x_j-coordinate in $\mathbb{R}^n$, and a corresponding partition P of S.

The crucial property of this partition is that each of the cells in P is entirely contained in one of the sets S_i. This proves the lemma, because the content $\nu(S)$ is equal to the content of all the cells in P, while the content $\nu(S_i)$ is equal to the sum of the contents of the cells in the partition P that are contained in S_i.

$\square$

From this simple fact, a much more interesting result is derived.

Definition 2 *A subset* $E \subseteq \mathbb{R}^n$ *is called* cellular *if it can be divided as a disjoint countable union of cells.*

Proposition 3 *Let* E *be a cellular set, and let* $R_1, R_2, R_3, \cdots$ *be a sequence of cells (not necessarily disjoint) that covers* E*, i.e.,*

$$E \subseteq R_1 \cup R_2 \cup R_3 \cup \cdots .$$

If $E = S_1 \cup S_2 \cup S_3 \cup \cdots$ *is any disjoint union of cells equal to* E*, then*

$$\sum_{i=1}^{\infty} \nu(S_i) \leq \sum_{j=1}^{\infty} \nu(R_j).$$

Remark. We allow for the possibility that the sum $\sum^{\infty} \nu(S_i)$ diverges. In that case the inequality in the proposition states that $\sum^{\infty} \nu(R_j)$ also diverges.

Proof. We first prove that the statement is true if $E = S$ is just a single set.

Let A be a cell which is slightly smaller than $E = S$, with $\nu(A) = (1-\varepsilon)\nu(S)$, and such that the compact closure $\overline{A}$ is contained in S. And let B_j be a cell which is slightly larger than R_j, with $\nu(B_j) = (1+\varepsilon)\nu(R_j)$, and such that the interior B_j^o contains R_j.

Then the open cover $\{B_j^o\}$ of the compact set $\overline{A}$ has a finite subcover, and we find that

$$B_1 \cup \cdots \cup B_M \supset B_1^o \cup \cdots \cup B_M^o \supset \overline{A} \supset A.$$

Because we only have a finite number of cells now, a partition argument shows that

$$\nu(B_1) + \cdots \nu(B_M) \geq \nu(A),$$

and so for every $\varepsilon > 0$, we have

$$(1+\varepsilon)\sum_{j=1}^{\infty} \nu(R_j) \geq (1-\varepsilon)\nu(S),$$

which gives the desired inequality.

For the general case $E = S_1 \cup S_2 \cup \cdots$, we argue by contradiction. If $\sum^{\infty} \nu(R_j) = \infty$, there is nothing to prove. So assume that this sum converges, and that

$$\sum_{i=1}^{\infty} \nu(S_i) > \sum_{j=1}^{\infty} \nu(R_j).$$

Then, by definition of the infinite sum

$$\sum_{i=1}^{\infty} \nu(S_i) = \lim_{N\to\infty} \sum_{i=1}^{N} \nu(S_i),$$

and because $\sum^{\infty} \nu(R_j)$ is finite, there is an integer N so that the partial sum satisfies

$$\sum_{i=1}^{N} \nu(S_i) > \sum_{j=1}^{\infty} \nu(R_j).$$

Define new cells (possibly empty)

$$C_{ij} = S_i \cap R_j.$$

Because the cells S_i are disjoint, the cells

$$C_{1j}, C_{2j}, \dots, C_{Nj}$$

are also disjoint. All of the cells C_{ij} with a fixed value for j are contained in the cell R_j. Therefore the previous lemma implies that

$$\nu(R_j) \geq \sum_{i=1}^{N} \nu(C_{ij}).$$

Therefore

$$\sum_{i=1}^{N} \nu(S_i) > \sum_{j=1}^{\infty} \nu(R_j) \geq \sum_{j=1}^{\infty} \left(\sum_{i=1}^{N} \nu(C_{ij}) \right) = \sum_{i=1}^{N} \left(\sum_{j=1}^{\infty} \nu(C_{ij}) \right).$$

This implies that for at least one value of $i = 1, \dots, N$ we must have

$$\nu(S_i) > \sum_{j=1}^{\infty} \nu(C_{ij}).$$

This inequality contradicts what we proved before if we can see that the cells C_{ij} with fixed i cover the cell S_i. This is true, because S_i is covered by the cells R_j. Every point $x \in S_i$ is contained in at least one R_j, but then also $x \in C_{ij} = S_i \cap R_j$. Therefore

$$S_i = \bigcup_{j=1}^{\infty} C_{ij}.$$

□

This result has important implications.

Corollary 4 *The total content of a cellular set $E = S_1 \cup S_2 \cup \cdots$, defined as*

$$\nu(E) = \nu(S_1) + \nu(S_2) + \cdots,$$

is independent of the way E is divided into disjoint cells.

Proof. If $E = R_1 \cup R_2 \cup \cdots$ is another disjoint union of cells, then $\bigcup S_i$ is covered by $\bigcup R_j$, and so $\sum \nu(S_i) \leq \sum \nu(R_j)$. But now $\bigcup R_j$ is also covered by $\bigcup S_i$, which gives $\sum \nu(R_j) \leq \sum \nu(S_i)$.

□

Thus, we have a well-defined notion of the content of cellular sets.