Problem 6.1.3 The easiest approach is to set up a $\int ..dy$ integral.

$$\int_0^1 [y^{1/2} - (y^2 - 1)]\, dy = \int_0^1 (y^{1/2} - y^2 + 1)\, dy$$
$$= \left[\frac{2}{3}y^{3/2} - \frac{1}{3}y^3 + y\right]_0^1 = [\frac{2}{3} - \frac{1}{3} + 1] - [0 - 0 + 0] = \frac{4}{3}.$$

Problem 6.2.22 The first solution uses the method of washers. The washers are vertical slices with thickness dx (perpendicular to the axis of rotation), so we work with x as the independent variable. Then the outer radius is $R = 1$, and the inner radius $r = 1 - x^3$. the volume of a single washer is

$$\pi(R^2 - r^2)\, dx = \pi[1 - (1 - x^3)^2]\, dx = \pi(2x^3 - x^6)\, dx.$$

Therefore, the total volume is the integral (summing all the washers)

$$V = \int_0^1 \pi(2x^3 - x^6)\, dx = \pi\left[\frac{2}{4}x^4 - \frac{1}{7}x^7\right]_0^1$$
$$= \pi[\frac{1}{2} - \frac{1}{7}] - \pi[0 - 0] = \frac{5}{14}\pi.$$

The alternative is to use the method of cylindrical shells. A *horizontal* line segment, parallel to the axis BC, generates a shell when rotated around the axis BC. Its thickness is dy, so we work with y as the independent variable. We first convert the formula for the curve into a y-function, replacing $y = x^3$ by $x = y^{1/3}$. The height and radius of a single shell are given as functions of y as follows,

$$h = 1 - y^{1/3},\ r = 1 - y.$$

Therefore, a single shell has volume

$$2\pi rh\, dy = 2\pi(1 - y)(1 - y^{1/3})dy = 2\pi(1 - y - y^{1/3} + y^{4/3})dy.$$

Summing all the cylindrical shells we obtain the integral

$$V = \int_0^1 2\pi(1 - y - y^{1/3} + y^{4/3})dy = 2\pi\left[y - \frac{1}{2}y^2 - \frac{3}{4}y^{4/3} + \frac{3}{7}y^{7/3}\right]_0^1$$
$$= 2\pi[1 - \frac{1}{2} - \frac{3}{4} + \frac{3}{7}] - 2\pi[0 - 0 - 0 + 0] = \frac{5}{14}\pi.$$

Multiple choice problems.

1. What is the total area of the region enclosed by the graphs of the functions $y = x^3$ and $y = 2x - x^3$?

   A.) 2/3 B.) 3/4 C.) 4/5 D.) 1 E.) 5/4 F.) 3/2

   The correct answer is "D".

   The points of intersection of the two graphs is obtained by solving $x^3 = 2x - x^3$. This gives $2x - 2x^3 = 0$, which can be factored as $2x(1 - x^2) = 0$. Hence $x = -1, 0, 1$ are the solutions, and the points of intersection are $(-1, -1), (0, 0), (1, 1)$.

   To complete a sketch of the graphs we need to determine which of the two has greater, and which has smaller y-values. Trying arbitrary values (like $x = -1/2, x = 1/2$) shows that for $-1 \le x \le 0$ we have $x^3 \ge 2x - x^3$, and on the interval $0 \le x \le 1$ we have $2x - x^3 \ge x^3$. (Now make a sketch.)

   Now we are ready to set up the integral. Following our analysis of the graphs, we see that the area-integral breaks up into two pieces, as follows,

$$\begin{aligned} A &= \int_{-1}^{0} [x^3 - (2x - x^3)]\,dx + \int_0^1 [(2x - x^3) - x^3]\,dx \\ &= \int_{-1}^{0} (2x^3 - 2x)\,dx + \int_0^1 (2x - 2x^3)\,dx. \end{aligned}$$

   Calculate the second integral,

$$\int_0^1 (2x - 2x^3)dx = \left[x^2 - \frac{2}{4}x^4\right]_0^1 = [1 - \frac{2}{4}] - [0 - 0] = \frac{1}{2}.$$

   The other integral has the same value, so the total area is $A = \frac{1}{2} + \frac{1}{2} = 1$.

2. What is the area of the region enclosed by the graph of the function $y = x^3$, the x-axis, and the tangent line to the graph of $y = x^3$ at the point $(1, 1)$?

   A.) 1/3 B.) 1/4 C.) 1/6 D.) 1/8 E.) 1/12 F.) 1/16

   The correct answer is "E".

   The slope of the tangent line at $(1, 1)$ is found by differentiating $y = x^3$. We get $y' = 3x^2$, which at $x = 1$ gives $y' = 3$. Thus the tangent line has equation $y = 3x + b$. Because $(1, 1)$ is on this line, it must be $y = 3x - 2$.

   To sketch the graphs, we must find the points of intersection of the x-axis with the function $y = x^3$ and the line $y = 3x - 2$. The two points are $(0, 0)$ (for the function itself) and $(\frac{2}{3}, 0)$ for the tangent line. (Now sketch the graph and the tangent line.)

   If we were to evaluate the area by means of a $\int ..dx$ integral, we would have to break it into two pieces: one integral for $\int_0^{2/3}$, and one for $\int_{2/3}^1$. However, it is easier to do this problem by means of a $\int ..dy$ integral. This means we must make y the independent variable, and so replace $y = x^3$ with $x = y^{1/3}$, and $y = 3x - 2$ with $x = \frac{1}{3}y + \frac{2}{3}$.

Because, from the point of view of the y-axis, the tangent line is 'above' the graph of $x = y^{1/3}$ (which you can only see in a sketch), the correct integral is,

$$A = \int_0^1 [(\frac{1}{3}y + \frac{2}{3}) - y^{1/3}]\, dy = \left[\frac{1}{6}y^2 + \frac{2}{3}y - \frac{3}{4}y^{4/3}\right]_0^1 = [\frac{1}{6} + \frac{2}{3} - \frac{3}{4}] - [0 + 0 - 0] = \frac{1}{12}.$$

3. What is the volume of the solid of revolution generated by rotating the region enclosed by the graphs of $y = x^3$ and $x = y^3$ between $x = 0$ and $x = 1$ around the axis of rotation $y = -1$?

   A.) $\frac{17}{14}\pi$ B.) $\frac{91}{70}\pi$ C.) $\frac{51}{35}\pi$ D.) $\frac{17}{15}\pi$ E.) $\frac{12}{7}\pi$ F.) $\frac{41}{28}\pi$

   The correct answer is "C".

   The two graphs intersect at the points $(0, 0)$ and $(1, 1)$. We'll use the method of 'washers', with vertical slicing. Hence the formula to use for the volume of a single washer is

   $$\pi(R^2 - r^2)dx.$$

   Since we work with an x-integral, we need to convert the relation $x = y^3$ into $y = x^{1/3}$. Inspection of s sketch of the graph shows that the outer radius of a washer is given by $R = x^{1/3} - (-1) = x^{1/3} + 1$, while the inner radius is $r = x^3 + 1$. Therefore, the formula for the volume of a single washer is,

   $$\begin{aligned} \pi[(x^{1/3} + 1)^2 - (x^3 + 1)^2]\, dx &= \pi[(x^{2/3} + 2x^{1/3} + 1) - (x^6 + 2x^3 + 1)]\, dx \\ &= \pi(x^{2/3} + 2x^{1/3} - x^6 - 2x^3)\, dx, \end{aligned}$$

   and integrating (summing all the washers) gives the total volume of the solid,

   $$\begin{aligned} \int_0^1 \pi(x^{2/3} + 2x^{1/3} - x^6 - 2x^3)\, dx &= \pi \left[\frac{3}{5}x^{5/3} + \frac{6}{4}x^{4/3} - \frac{1}{7}x^7 - \frac{2}{4}x^4\right]_0^1 \\ &= \pi(\frac{3}{5} + \frac{6}{4} - \frac{1}{7} - \frac{2}{4}) - \pi(0 + 0 - 0 - 0) = \frac{51}{35}\pi. \end{aligned}$$