Solutions to homework 3

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Remark: I will be labelling questions $c.s.q$, where c is the chapter number; s is the section number; and q is the question number.

Question 7.4.70: Evaluate the integral

$$\int_e^6 \frac{dx}{x\ln(x)}.$$

Solution: This question requires a substitution. Let $u = \ln(x)$. Then $du = \frac{1}{x}dx$. Also when $x = e$, then $u = 1$ and when $x = 6$, $u = \ln(6)$. Thus,

$$\begin{aligned}\int_e^6 \frac{dx}{x\ln(x)} &= \int_1^{\ln(6)} \frac{du}{u} \\ &= [\ln(u)]_1^{\ln(6)} \\ &= \ln(\ln(6)).\end{aligned}$$

Question 7.5.63: Evaluate the integral

$$\int_0^{\frac{1}{2}} \frac{\sin^{-1}(x)dx}{\sqrt{1-x^2}}.$$

Solution: This question also wants a substitution. Let $u = \sin^{-1}(x)$. Then $du = \frac{dx}{\sqrt{1-x^2}}$. Also when $x = 0$, $u = 0$ and when $x = \frac{1}{2}$, $u = \frac{\pi}{6}$. Thus,

$$\begin{aligned}\int_0^{\frac{1}{2}} \frac{\sin^{-1}(x)dx}{\sqrt{1-x^2}} &= \int_0^{\frac{\pi}{6}} udu \\ &= \frac{\pi^2}{72}.\end{aligned}$$

Multiple choice question 1: Evaluate the limit

$$\lim_{x\to\infty} \cos^{-1}\left(\frac{1+x}{\sqrt{3}+x\sqrt{2}}\right).$$

Solution:

$$\lim_{x\to\infty} \cos^{-1}\left(\frac{1+x}{\sqrt{3}+x\sqrt{2}}\right) = \lim_{x\to\infty} \cos^{-1}\left(\frac{\frac{1}{x}+1}{\frac{\sqrt{3}}{x}+\sqrt{2}}\right).$$

Here $\frac{1}{x} \to 0$ as $x \to \infty$, so

$$\begin{aligned}\lim_{x\to\infty} \cos^{-1}\left(\frac{1+x}{\sqrt{3}+x\sqrt{2}}\right) &= \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) \\ &= \frac{\pi}{4}.\end{aligned}$$

Multiple choice question 2: Evaluate

$$\int_0^2 \frac{tdt}{\sqrt{16-t^4}}.$$

Solution: Let $u = t^2$. Then $tdt = \frac{1}{2}du$. Also, when $t = 0$, $u = 0$ and when $t = 2$, $u = 4$. Thus

$$\int_0^2 \frac{tdt}{\sqrt{16-t^4}} = \int_0^4 \frac{\frac{1}{2}du}{\sqrt{16-u^2}}.$$

Now let $v = \frac{u}{4}$. Then $4dv = du$. Also, when $u = 0$, $v = 0$ and when $u = 4$, $v = 1$. Thus

$$\begin{aligned} \int_0^4 \frac{\frac{1}{2}du}{\sqrt{16-u^2}} &= \int_0^1 \frac{dv}{2\sqrt{1-v^2}} \\ &= \frac{1}{2}\left[\sin^{-1}(v)\right]_0^1 \\ &= \frac{\pi}{4}. \end{aligned}$$

Multiple choice question 3: Let $f(x) = \sinh(x^2 + 2x + 1)$. Find $f'(0)$.

Solution: Using the chain-rule we have

$$f'(x) = \cosh(x^2 + 2x + 1)(2x + 2).$$

Thus $f'(0) = 2\cosh(0) = 2$.