

Solutions to homework 4

By Håkan Nordgren

REMARK: Again, I will be labelling questions *c.s.q.*, where *c* is the chapter number, *s* is the section number, and *q* is the question number.

Question 8.1.22: Evaluate the integral

$$\int t^{\frac{1}{2}} \ln(t) dt.$$

Solution: We know how to differentiate $\ln(t)$, but it is not as clear how to integrate it, so it is usually best, in cases such as this one to take $f(t) = \ln(t)$ and $\left(\frac{dg}{dt}\right)(t) = t^{\frac{1}{2}}$. Then $\left(\frac{df}{dt}\right)(t) = \frac{1}{t}$ and $g(t) = \frac{2}{3}t^{\frac{3}{2}}$. Thus,

$$\begin{aligned} \int t^{\frac{1}{2}} \ln(t) dt &= \ln(t) \frac{2}{3} t^{\frac{3}{2}} - \frac{2}{3} \int t^{\frac{1}{2}} dt \\ &= \ln(t) \frac{2}{3} t^{\frac{3}{2}} - \frac{4}{9} t^{\frac{3}{2}} + C. \end{aligned}$$

Question 8.2.2: Evaluate the integral

$$\int \sin^6(x) \cos^3(x) dx.$$

Solution: $\cos^2(x) = 1 - \sin^2(x)$, so we can write

$$\int \sin^6(x) \cos^3(x) dx = \int \sin^6(x) \cos(x) dx - \int \sin^8(x) \cos(x) dx.$$

In both integrals make the change of variable $u = \sin(x)$. Then $du = \cos(x) dx$ and

$$\begin{aligned} \int \sin^6(x) \cos^3(x) dx &= \int u^6 du - \int u^8 du \\ &= \frac{1}{7} u^7 - \frac{1}{9} u^9 \\ &= \frac{1}{7} \sin^7(x) - \frac{1}{9} \sin^9(x). \end{aligned}$$

Multiple choice question 1: Evaluate the limit

$$\lim_{x \rightarrow \infty} \left(\frac{1}{2} \ln(x^2 + 1) - \ln(2x + 1) \right).$$

Solution:

$$\begin{aligned}\lim_{x \rightarrow \infty} \left(\frac{1}{2} \ln(x^2 + 1) - \ln(2x + 1) \right) &= \lim_{x \rightarrow \infty} \frac{1}{2} \ln \left(\frac{x^2 + 1}{(2x + 1)^2} \right) \\ &= \lim_{x \rightarrow \infty} \frac{1}{2} \ln \left(\frac{x^2 + 1}{4x^2 + 4x + 1} \right) \\ &= \\ &= \lim_{x \rightarrow \infty} \frac{1}{2} \ln \left(\frac{1 + \frac{1}{x^2}}{4 + \frac{4}{x} + \frac{1}{x^2}} \right) \\ &= \frac{1}{2} \ln \left(\frac{1}{4} \right) \\ &= \ln \left(\frac{1}{2} \right) \\ &= -\ln(2),\end{aligned}$$

which is answer-choice B.

Multiple choice question 2: Evaluate the limit

$$\lim_{x \rightarrow 0} x^2 \ln(x^2).$$

Solution:

$$\lim_{x \rightarrow 0} x^2 \ln(x^2) = \lim_{x \rightarrow 0} \frac{\ln(x^2)}{\frac{1}{x^2}}.$$

By using L'Hopital's rule, we have

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\ln(x^2)}{\frac{1}{x^2}} &= \lim_{x \rightarrow 0} \frac{\frac{1}{x^2} 2x}{\frac{-2}{x^3}} \\ &= \lim_{x \rightarrow 0} \frac{\frac{2}{x}}{\frac{-2}{x^3}} \\ &= \lim_{x \rightarrow 0} \frac{-2x^3}{2x} \\ &= \lim_{x \rightarrow 0} -x^2, \\ &= 0,\end{aligned}$$

which is answer-choice B again.

Multiple choice question 3: Evaluate the integral

$$\int_0^1 e^{(x^{\frac{1}{3}})} dx.$$

Solution: If we make the substitution $u = x^{\frac{1}{3}}$, then $du = \frac{1}{3}x^{-\frac{2}{3}} dx = \frac{1}{3}u^{-2} dx$. Thus, $dx = 3u^2 du$. Also, when $x = 0$, $u = 0$, and when $x = 1$, $u = 1$. Thus the limits are left unchanged. This yields

$$\int_0^1 e^{(x^{\frac{1}{3}})} dx = 3 \int_0^1 e^u u^2 du.$$

This set-up begs to be integrated by parts. Twice. We have to integrate $\int_0^1 3e^u u^2 du$, so let $f(u) = u^2$ and let $\left(\frac{dg}{dx}\right)(u) = 3e^u$. Then

$$\int_0^1 3e^u u^2 du = [u^2 3e^u]_0^1 - \int_0^1 6e^u u du.$$

Now let $f(u) = u$ and let $\left(\frac{dg}{dx}\right)(u) = 6e^u$. Then

$$\begin{aligned} - \int_0^1 6e^u u du &= -[u 6e^u]_0^1 + \int_0^1 6e^u du \\ &= -6e + 6[e^u]_0^1 \\ &= -6e + 6e - 6 \\ &= -6. \end{aligned}$$

The grand-total is thus,

$$\begin{aligned} \int_0^1 \frac{1}{3} e^u u^2 du &= [u^2 3e^u]_0^1 - 6 \\ &= 3e - 6, \end{aligned}$$

which is answer choice D.

Multiple choice question 3: Evaluate the integral

$$\int_0^{\frac{\pi}{4}} \frac{\tan^2(x)}{\cos^4(x)} dx.$$

Solution: If we make the substitution $u = \tan(x)$, then $du = \sec^2(x) dx$. Also, when $x = 0$, $u = 0$, and when $x = \frac{\pi}{4}$, $u = 1$. Thus the change of variables gives

$$\begin{aligned} \int_0^{\frac{\pi}{4}} \frac{\tan^2(x)}{\cos^4(x)} dx &= \int_0^1 u^2 \sec^2(x) \sec^2(x) dx \\ &= \int_0^1 u^2 \sec^2(x) du. \end{aligned}$$

Using the identity, $\tan^2(x) + 1 = \sec^2(x)$, we have

$$\begin{aligned} \int_0^1 u^2 \sec^2(x) du &= \int_0^1 u^2 (1 + u^2) du \\ &= \int_0^1 (u^2 + u^4) du \\ &= \left[\frac{1}{3} u^3 + \frac{1}{5} u^5 \right]_0^1 \\ &= \frac{1}{3} + \frac{1}{5} \\ &= \frac{8}{15}, \end{aligned}$$

which is answer-choice B.