

Solutions to some problems from homework set 6 (sections 8.4, 8.5, 8.8, 8.8)

Problem 8.5.4 We need to solve the integral

$$\int \frac{x}{\sqrt{3-x^4}} dx$$

This is not an integral covered in sections 8.1-8.4. But with $3-x^2$ instead of $3-x^4$ under the square root, the integral would look familiar. By substituting $u = x^2$, $du = 2x dx$ we achieve the desired result. We get

$$\int \frac{x}{\sqrt{3-x^4}} dx = \frac{1}{2} \int \frac{1}{\sqrt{3-u^2}} du = \frac{1}{2} \sin^{-1} \frac{u}{\sqrt{3}} = \frac{1}{2} \sin^{-1} \frac{x^2}{\sqrt{3}}.$$

Problem 8.8.38 Evaluate

$$\int_0^2 \frac{x-3}{2x-3} dx.$$

You must notice that this is an *improper integral*, because the denominator $2x-3$ equals 0 if $x = 3/2$, which falls in the interval of integration $[0, 2]$. We can solve this integral by means of the substitution $u = 2x-3$, $du = 2dx$. Then $x = \frac{1}{2}u + \frac{3}{2}$, and therefore $x-3 = \frac{1}{2}u - \frac{3}{2}$. We obtain,

$$\int_0^2 \frac{x-3}{2x-3} dx = \int_{-3}^1 \frac{u-3}{2u} du = \int_{-3}^1 \left(\frac{1}{2} - \frac{3}{2u} \right) du = 2 - \int_{-3}^1 \frac{3}{2u} du.$$

To evaluate the improper integral, we must break it into two parts,

$$\int_{-3}^1 \frac{3}{2u} du = \int_{-3}^0 \frac{3}{2u} du + \int_0^1 \frac{3}{2u} du.$$

Both of these integrals diverge. for example,

$$\int_{-3}^0 \frac{3}{2u} du = \lim_{t \uparrow 0^-} \left[\frac{3}{2} \ln |u| \right]_{-3}^t = \frac{3}{2} \ln 3 - \lim_{t \uparrow 0^-} \frac{3}{2} \ln |t|,$$

which diverges because $\lim_{t \uparrow 0^-} \ln |t| = -\infty$. A similar calculation shows that \int_0^1 diverges as well.

Multiple choice problems

1. Evaluate the following integral

$$\int_2^3 \frac{1}{(x^2 - 4x + 5)^2} dx.$$

- A.) $\frac{1}{2}\pi + \frac{1}{2}$ B.) $\frac{1}{2}\pi + \frac{1}{4}$ C.) $\frac{1}{4}\pi + \frac{1}{4}$ D.) $\frac{1}{4}\pi + \frac{1}{8}$ E.) $\frac{1}{8}\pi + \frac{1}{4}$ F.) $\frac{1}{8}\pi + \frac{1}{8}$

Solution. Based on a completion of the square

$$x^2 - 4x + 5 = (x-2)^2 + 1,$$

we try the substitution $u = x - 2$, $du = dx$. This changes the integral to

$$\int_0^1 \frac{1}{(u^2 + 1)^2} du.$$

Note that the new boundaries of the integral are $u = 0$ (for $x = 2$) and $u = 1$ (for $x = 3$). (For definite integrals it is usually easier to change the values of the boundaries at the moment you do the substitution. This eliminates the need to reverse the substitution at the end.) The antiderivative can be found in the Table of Standard Integrals, and we find

$$\left[\frac{u}{2(u^2 + 1)} + \frac{1}{2} \tan^{-1} u \right]_0^1 = \frac{1}{4} + \frac{1}{2} \tan^{-1} 1 = \frac{1}{4} + \frac{\pi}{8}.$$

The correct answer is E.)

2. Evaluate the following integral

$$\int_2^3 \frac{x^3 + 1}{x^3 - x^2} dx.$$

A.) $\frac{5}{6} + \ln \frac{8}{3}$ B.) $\frac{4}{5} + \ln \frac{7}{3}$ C.) $\frac{5}{7} + \ln \frac{7}{4}$ D.) $\frac{6}{7} + \ln \frac{8}{5}$ E.) $\frac{2}{3} + \ln \frac{11}{12}$ F.) $\frac{3}{2} + \ln \frac{3}{4}$

Solution. A long division shows that

$$\frac{x^3 + 1}{x^3 - x^2} = 1 + \frac{x^2 + 1}{x^3 - x^2}.$$

So the first step is simply,

$$\begin{aligned} \int_2^3 \frac{x^3 + 1}{x^3 - x^2} dx &= \int_2^3 \left[1 + \frac{x^2 + 1}{x^3 - x^2} \right] dx \\ &= [x]_2^3 + \int_2^3 \frac{x^2 + 1}{x^3 - x^2} dx \\ &= 1 + \int_2^3 \frac{x^2 + 1}{x^2(x-1)} dx. \end{aligned}$$

We factored the denominator, in preparation for the next step. Following the rules for partial fractions, we must solve

$$\frac{x^2 + 1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} = \frac{Ax(x-1) + B(x-1) + Cx^2}{x^2(x-1)}.$$

Concentrating on the numerators we get

$$Ax(x-1) + B(x-1) + Cx^2 = x^2 + 1.$$

This equation must hold for all values of x . One way to solve this is to cleverly choose some values for x , and see what happens. With $x = 0$ we find $A0 - B + C0 = -B = 0^2 + 1 = 1$, and so $B = -1$. Next, $x = 1$ gives $A0 + B0 + C = C = 1^2 + 1 = 2$, or $C = 2$. Finally, we can take $x = 2$ (a random value), and find $2A + B + 4C = 2^2 + 1 = 5$. With $B = -1$, $C = 2$ this becomes $2A + 7 = 5$, or $A = -1$. Putting it all together we get

$$\frac{x^2 + 1}{x^2(x-1)} = \frac{-1}{x} + \frac{-1}{x^2} + \frac{2}{x-1}.$$

The indefinite integral is therefore

$$\int \frac{-1}{x} dx + \int \frac{-1}{x^2} dx + \int \frac{2}{x-1} dx = -\ln x + \frac{1}{x} + 2 \ln(x-1).$$

Finally, the definite integral we need is

$$\begin{aligned} 1 + \int_2^3 \frac{x^2 + 1}{x^2(x-1)} dx &= 1 + \left[-\ln x + \frac{1}{x} + 2 \ln(x-1) \right]_2^3 \\ &= 1 + \left[-\ln 3 + \frac{1}{3} + 2 \ln 2 \right] - \left[-\ln 2 + \frac{1}{2} + 2 \ln 1 \right] \\ &= 1 + \frac{1}{3} - \frac{1}{2} - \ln 3 + 3 \ln 2 = \frac{5}{6} - \ln 3 + \ln 2^3 = \frac{5}{6} + \ln \frac{2^3}{3}. \end{aligned}$$

The correct answer is A.)

3. Evaluate the following improper integral,

$$\int_1^{\infty} \frac{\tan^{-1} x}{x^2} dx.$$

- A.) $4\pi - \ln 8$ B.) $2\pi - \ln 8$ C.) $\pi + \ln 4$ D.) $\frac{1}{2}\pi + \ln 2$ E.) $\frac{1}{4}\pi + \frac{1}{2}\ln 2$
 F.) The integral diverges

Solution. The first step is to solve the indefinite integral. We use integration by parts and take $u = \tan^{-1} x$ and $dv = x^{-2} dx$. Integrating this latter expression, we find $v = -x^{-1}$, while $du = \frac{1}{x^2+1}$. Thus,

$$\int u dv = uv - \int v du = -x^{-1} \tan^{-1} x + \int \frac{1}{x(x^2+1)} dx.$$

The integral on the right hand side of this equation must be broken up into partial fractions.

$$\frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} = \frac{A(x^2+1) + (Bx+C)x}{x(x^2+1)},$$

leads to

$$A(x^2+1) + Bx^2 + Cx = 1.$$

Picking $x = 0$ gives $A = 1$. Trying $x = 1$ and $x = -1$, respectively, we find $2A + B + C = 1$ and $2A + B - C = 1$. We know that $A = 1$, so we get $2 + B + C = 1$ and $2 + B - C = 1$. If you add these equations you get $4 + 2B = 2$, or $B = -1$. Then, finally, $C = 0$. Thus,

$$\int \frac{1}{x(x^2+1)} dx = \int \frac{1}{x} dx + \int \frac{-x}{x^2+1} dx = \ln x - \frac{1}{2} \ln(x^2+1).$$

This last integral was obtained by means of the substitution $u = x^2 + 1$, $du = 2x dx$, as follows,

$$\int \frac{-x}{x^2+1} dx = -\frac{1}{2} \int \frac{1}{u} du = -\frac{1}{2} \ln u = -\frac{1}{2} \ln(x^2+1).$$

The end result is the solution of the indefinite integral

$$\int \frac{\tan^{-1} x}{x^2} dx = -\frac{\tan^{-1} x}{x} + \ln x - \frac{1}{2} \ln(x^2+1) = -\frac{\tan^{-1} x}{x} + \ln \frac{x}{\sqrt{x^2+1}}.$$

We have combined the logarithmic terms into one, because that will make it easier to evaluate the limits in the final step, where we determine the improper integral,

$$\begin{aligned} \int_1^{\infty} \frac{\tan^{-1} x}{x^2} dx &= \lim_{t \rightarrow \infty} \left[-\frac{\tan^{-1} x}{x} + \ln \frac{x}{\sqrt{x^2+1}} \right]_1^t \\ &= \lim_{t \rightarrow \infty} \left[-\frac{\tan^{-1} t}{t} + \ln \frac{t}{\sqrt{t^2+1}} \right] - \left[-\tan^{-1} 1 + \ln \frac{1}{\sqrt{2}} \right]. \end{aligned}$$

We evaluate the limits one by one. First, $\lim_{t \rightarrow \infty} \tan^{-1} t = \pi/2$, and so

$$\lim_{t \rightarrow \infty} -\frac{\tan^{-1} t}{t} = 0.$$

Next, because

$$\lim_{t \rightarrow \infty} \frac{t}{\sqrt{t^2 + 1}} = \lim_{t \rightarrow \infty} \frac{t}{\sqrt{t^2}} = \lim_{t \rightarrow \infty} \frac{t}{t} = 1,$$

we get

$$\lim_{t \rightarrow \infty} \ln \frac{t}{\sqrt{t^2 + 1}} = \ln 1 = 0.$$

All that is left is therefore

$$\int_1^{\infty} \frac{\tan^{-1} x}{x^2} dx = \tan^{-1} 1 - \ln \frac{1}{\sqrt{2}} = \frac{\pi}{4} + \ln \sqrt{2} = \frac{\pi}{4} + \frac{1}{2} \ln 2.$$

The correct answer is E.)

4. We know that

$$\int_0^1 \frac{1}{x^2 + 1} dx = \tan^{-1} 1 = \frac{\pi}{4}.$$

Use *Simpson's Rule* with $n = 4$ to find an approximate value for the integral (i.e., compute S_4), and then multiply S_4 by 4 to get an approximate value for π . Use a calculator.

If you know that the first eight digits of π are 3.1415926..., then how many of these digits are correct in your approximation? (Example: if you found that $4S_4 = 3.1427\dots$, then you would have 3 correct digits of π .)

A.) 3 B.) 4 C.) 5 D.) 6 E.) 7 F.) 8

Solution. We divide the interval $[0, 1]$ in four equal subintervals by taking $x_0 = 0, x_1 = \frac{1}{4}, x_2 = \frac{2}{4}, x_3 = \frac{3}{4}, x_4 = 1$. Clearly, the width of each subinterval is $\Delta x = \frac{1}{4}$. Writing

$$f(x) = \frac{1}{x^2 + 1},$$

the Simpson rule gives,

$$\begin{aligned} S_4 &= \frac{\Delta x}{3} (f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)) \\ &= \frac{1}{12} \left(1 + 4 * \frac{16}{17} + 2 * \frac{4}{5} + 4 * \frac{16}{25} + \frac{1}{2} \right) \\ &= \frac{8011}{10200} = 0.78539215\dots \end{aligned}$$

This is an approximation for the integral, which equals $\pi/4$. To get an approximation for π , we multiply by 4 and find

$$4S_4 = \frac{8011}{2550} = 3.1415686\dots$$

Thus, we have 5 correct decimals.

The correct answer is C.)