

Math 104
Answers for Midterm Exam 2
March 14, 2007

1. Evaluate the integral.

$$\int_0^2 \frac{x^2}{\sqrt{x^3+1}} dx$$

A.) $8/9$ B.) $4/3$ C.) $52/9$ D.) $\frac{2\sqrt{2}}{3}$ E.) $\frac{4\sqrt{2}}{9}$ F.) $\sqrt{5}-1$

Answer. We first use u -substitution, namely $u = x^3+1$, so $du = 3x^2 dx$, and $\frac{1}{3} du = x^2 dx$. This yields

$$\int \frac{x^2}{\sqrt{x^3+1}} dx = \frac{1}{3} \int \frac{1}{\sqrt{u}} du = \left(\frac{1}{3}\right) \left(\frac{u^{1/2}}{1/2}\right) + C = \frac{2}{3}\sqrt{u} + C = \frac{2}{3}\sqrt{x^3+1} + C$$

Now we consider the limits of integration. We have

$$\int_0^2 \frac{x^2}{\sqrt{x^3+1}} dx = \frac{2}{3}\sqrt{x^3+1} \Big|_{x=0}^2 = \frac{2}{3} \left(\sqrt{2^3+1} - \sqrt{0^3+1} \right) = \frac{2}{3}(3-1) = \frac{4}{3}$$

So the correct answer is **B**, namely, $4/3$.

2. Evaluate the integral.

$$\int_0^1 x \sinh x \, dx$$

A.) 0

B.) 1

C.) $\sinh 1$

D.) $\cosh 1$

E.) $\cosh 1 - \sinh 1$

F.) $\sinh 1 - \cosh 1$

Answer. We use integration by parts, writing $u = x$ and $dv = \sinh x$. This gives us $du = dx$ and $v = \cosh x$. So we have

$$\int x \sinh x \, dx = x \cosh x - \int \cosh x \, dx = x \cosh x - \sinh x + C$$

So we conclude

$$\int_0^1 x \sinh x \, dx = (x \cosh x - \sinh x)|_{x=0}^1 = (1 \cosh 1 - \sinh 1) - (0 \cosh 0 - \sinh 0) = \cosh 1 - \sinh 1$$

So the correct answer is **E**, namely, $\cosh 1 - \sinh 1$.

3. Evaluate the integral.

$$\int_3^4 \sqrt{-x^2 + 6x - 8} dx$$

A.) 0 B.) $\pi/6$ C.) $\pi/4$ D.) $\pi/3$ E.) $\pi/2$ F.) π

Answer. We first complete the square. We have

$$x^2 - 6x + 9 = (x - 3)^2$$

Subtracting “1” on both sides yields

$$x^2 - 6x + 8 = (x - 3)^2 - 1$$

and thus

$$-x^2 + 6x - 8 = 1 - (x - 3)^2$$

Therefore

$$\int \sqrt{-x^2 + 6x - 8} dx = \int \sqrt{1 - (x - 3)^2} dx = \int \sqrt{1 - u^2} du$$

where $u = x - 3$ and $du = dx$. Then we can use u -substitution, writing $u = \sin \theta$ and $du = \cos \theta d\theta$. We obtain

$$\int \sqrt{1 - u^2} du = \int \sqrt{1 - \sin^2 \theta} \cos \theta d\theta = \int \sqrt{\cos^2 \theta} \cos \theta d\theta = \int \cos^2 \theta d\theta$$

Using a half-angle identity, we have

$$\int \cos^2 \theta d\theta = \int \frac{1}{2}(1 + \cos 2\theta) dx = \frac{1}{2} \left(\theta + \frac{1}{2} \sin 2\theta \right) + C = \frac{1}{2} (\sin^{-1}(u) + \sin \theta \cos \theta) + C$$

We know $\sin \theta = \frac{u}{1} = x - 3$, so $\cos \theta = \sqrt{1 - u^2} = \sqrt{1 - (x - 3)^2}$. So

$$\frac{1}{2} (\sin^{-1}(u) + \sin \theta \cos \theta) = \frac{1}{2} (\sin^{-1}(x - 3) + (x - 3)\sqrt{1 - (x - 3)^2})$$

We conclude

$$\int_3^4 \sqrt{-x^2 + 6x - 8} dx = \frac{1}{2} (\sin^{-1}(x - 3) + (x - 3)\sqrt{1 - (x - 3)^2}) \Big|_{x=3}^4 = \frac{1}{2} \sin^{-1}(1) = \frac{\pi}{4}$$

So the correct answer is **C**, namely, $\pi/4$.

4. Evaluate the integral.

$$\int_0^1 \frac{x^2 - 1}{x^2 + 1} dx$$

A.) 0 B.) 1 C.) $1 - \frac{\pi}{2}$ D.) $-1/2$ E.) $1 + \frac{\pi}{4}$ F.) $1/4$

Answer. We first use long division, which yields

$$\frac{x^2 - 1}{x^2 + 1} = 1 - \frac{2}{x^2 + 1}$$

and therefore

$$\int \frac{x^2 - 1}{x^2 + 1} dx = \int \left(1 - \frac{2}{x^2 + 1} \right) dx = x - 2 \tan^{-1}(x) + C$$

So we conclude

$$\int_0^1 \frac{x^2 - 1}{x^2 + 1} dx = (1 - 2 \tan^{-1}(1)) - (0 - 2 \tan^{-1}(0)) = 1 - \frac{\pi}{2}$$

So the correct answer is **C**, namely, $1 - \frac{\pi}{2}$.

5. Evaluate the integral.

$$\int_0^{\pi/6} \frac{\tan^4(2x)}{\cos^2(2x)} dx$$

A.) $\frac{9\sqrt{3}}{10}$ B.) $\frac{9\sqrt{3}}{16} - \frac{\pi}{4}$ C.) $\frac{\sqrt{3}}{270}$ D.) $-\frac{9\sqrt{3}}{128} + \frac{\pi}{16}$ E.) $\frac{\sqrt{3}}{2} - \frac{\pi}{6}$ F.) $-\frac{\sqrt{3}}{16} + \frac{\pi}{12}$

Answer. We write

$$\int \frac{\tan^4(2x)}{\cos^2(2x)} dx = \int \tan^4(2x) \sec^2(2x) dx$$

Then we use a u -substitution, writing $u = \tan(2x)$ and $du = 2 \sec^2(2x) dx$, so $\frac{1}{2} du = \sec^2(2x) dx$. So we have

$$\int \tan^4(2x) \sec^2(2x) dx = \frac{1}{2} \int u^4 du = \frac{u^5}{10} + C = \frac{\tan^5(2x)}{10} + C$$

$$\int_0^{\pi/6} \frac{\tan^4(2x)}{\cos^2(2x)} dx = \left. \frac{\tan^5(2x)}{10} \right|_{x=0}^{\pi/6} = \frac{\tan^5(\pi/3)}{10} - \frac{\tan^5(0)}{10} = \frac{(\sqrt{3})^5}{10} = \frac{9\sqrt{3}}{10}$$

So the correct answer is **A**, namely, $\frac{9\sqrt{3}}{10}$.

6. Evaluate the integral.

$$\int_1^2 \frac{x-1}{x^2+x} dx$$

A.) $\ln \frac{9}{8}$

B.) $\ln \frac{9}{4}$

C.) $\ln \frac{9}{2}$

D.) $\ln 2$

E.) $\ln 4$

F.) $\ln 9$

Answer. We use partial fractions. We write

$$\frac{x-1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1} = \frac{A(x+1) + Bx}{x(x+1)}$$

so $x-1 = A(x+1) + Bx = (A+B)x + A$, and thus $A = -1$ and $B = 2$. So $\frac{x-1}{x(x+1)} = \frac{-1}{x} + \frac{2}{x+1}$. It follows that $\int \frac{x-1}{x(x+1)} dx = \int \frac{-1}{x} dx + 2 \int \frac{1}{x+1} dx = -\ln|x| + 2\ln|x+1| + C$. In summary,

$$\int \frac{x-1}{x^2+x} dx = -\ln|x| + 2\ln|x+1| + C$$

We conclude

$$\int_1^2 \frac{x-1}{x^2+x} dx = (-\ln 2 + 2\ln 3) - (-\ln 1 + 2\ln 2) = -3\ln 2 + 2\ln 3 = \ln(9/8)$$

So the correct answer is **A**, namely, $\ln \frac{9}{8}$.

7. Determine whether the integral is convergent or divergent. Evaluate the integral, if the integral is convergent.

$$\int_0^{\infty} \frac{x}{(x^2 + 4)^2} dx$$

A.) 0 B.) 1/2 C.) 1/4 D.) 1/8 E.) 1/16 F.) The integral diverges

Answer. We first use u -substitution, namely $u = x^2 + 4$, so $du = 2x dx$, and $\frac{1}{2} du = x dx$. This yields

$$\int \frac{x}{(x^2 + 4)^2} dx = \frac{1}{2} \int \frac{1}{u^2} du = \left(\frac{1}{2}\right) \left(\frac{u^{-1}}{-1}\right) + C = -\frac{1}{2u} + C = -\frac{1}{2(x^2 + 4)}$$

Now we consider the limits of integration. We have

$$\int_0^{\infty} \frac{x}{(x^2 + 4)^2} dx = \lim_{b \rightarrow \infty} \int_0^b \frac{x}{(x^2 + 4)^2} dx = \lim_{b \rightarrow \infty} \left. -\frac{1}{2(x^2 + 4)} \right|_{x=0}^b$$

Equivalently,

$$\int_0^{\infty} \frac{x}{(x^2 + 4)^2} dx = \lim_{b \rightarrow \infty} \left[\left(-\frac{1}{2(b^2 + 4)} \right) - \left(-\frac{1}{2(0^2 + 4)} \right) \right] = 0 + \frac{1}{2(4)} = \frac{1}{8}$$

So the correct answer is **D**, namely, 1/8.

8. Find the length of the curve

$$y = \frac{1}{4}x^2 - \frac{1}{2} \ln x, \quad 1 \leq x \leq e.$$

A.) $\frac{e^3}{12} - \frac{e}{2} - \frac{1}{4e} + \frac{2}{3}$

B.) $\frac{e^3}{12} + \frac{e}{2} - \frac{1}{4e} - \frac{1}{3}$

C.) $\frac{e^2}{4} + \frac{1}{4}$

D.) $\frac{e^2}{4} - \frac{3}{4}$

E.) $\frac{e^3}{12} + \frac{5}{12}$

F.) $\frac{e^3}{12} - \frac{7}{12}$

Answer. We notice that $\frac{dy}{dx} = (\frac{1}{2}x - \frac{1}{2x})$, so $(\frac{dy}{dx})^2 = (\frac{1}{2}x - \frac{1}{2x})^2 = \frac{1}{4}x^2 - \frac{1}{2} + \frac{1}{4x^2}$. This gives us

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{1}{4}x^2 - \frac{1}{2} + \frac{1}{4x^2} = \frac{1}{4}x^2 + \frac{1}{2} + \frac{1}{4x^2} = \left(\frac{1}{2}x + \frac{1}{2x}\right)^2$$

Thus, the arc length is

$$L = \int_1^e \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_1^e \sqrt{\left(\frac{1}{2}x + \frac{1}{2x}\right)^2} dx = \int_1^e \frac{1}{2}x + \frac{1}{2x} dx = \left(\frac{x^2}{4} + \frac{1}{2} \ln x\right) \Big|_{x=1}^e$$

Simplifying, this yields

$$L = \left(\frac{e^2}{4} + \frac{1}{2} \ln e\right) - \left(\frac{1}{4} + \frac{1}{2} \ln 1\right) = \frac{e^2}{4} + \frac{1}{2} - \frac{1}{4} = \frac{e^2}{4} + \frac{1}{4}$$

So the correct answer is **C**, namely, $\frac{e^2}{4} + \frac{1}{4}$.

9. Find the area of the surface obtained by rotating the curve

$$y = x^3, \quad 0 \leq x \leq 1$$

about the x -axis.

A.) $\pi/27$

B.) $4\pi/15$

C.) $\pi/3$

D.) $\frac{\pi}{27} (10\sqrt{10} - 1)$

E.) $\frac{4}{15}\pi(1 + \sqrt{2})$

F.) $\frac{\pi}{3} (2\sqrt{2} - 1)$

Answer. We notice that $\frac{dy}{dx} = 3x^2$, so $1 + \left(\frac{dy}{dx}\right)^2 = 1 + 9x^4$. So the surface area is

$$\int_0^1 2\pi x^3 \sqrt{1 + 9x^4} dx$$

Using u -substitution with $u = 1 + 9x^4$ and $du = 36x^3 dx$, we have $\frac{1}{36} du = x^3 dx$, so we obtain

$$\int 2\pi x^3 \sqrt{1 + 9x^4} dx = \frac{2\pi}{36} \int \sqrt{u} du = \left(\frac{\pi}{18}\right) \left(\frac{u^{3/2}}{3/2}\right) + C = \frac{\pi}{27} (1 + 9x^4)^{3/2} + C$$

So the surface area is

$$\int_0^1 2\pi x^3 \sqrt{1 + 9x^4} dx = \frac{\pi}{27} (1 + 9x^4)^{3/2} \Big|_{x=0}^1 = \frac{\pi}{27} (10\sqrt{10} - 1)$$

So the correct answer is **D**, namely, $\frac{\pi}{27} (10\sqrt{10} - 1)$.

10. The length of a phone call in minutes is an exponential random variable with probability density function

$$f(t) = \begin{cases} 0 & \text{if } t < 0, \\ \frac{1}{10}e^{-t/10} & \text{if } t \geq 0, \end{cases}$$

with the time t measured in minutes.

Find the probability that a phone call is longer than 10 minutes.

A.) $1/e$ B.) $1/10$ C.) $9/10$ D.) $\frac{1}{10}e^{-1/10}$ E.) $e^{-1/10}$ F.) $1 - e^{-1/10}$

Answer. The probability that a phone call is longer than 10 minutes is

$$\int_{10}^{\infty} \frac{1}{10} e^{-t/10} dt$$

We use u -substitution, writing $u = -t/10$, so $du = -\frac{1}{10} dt$, and $-du = \frac{1}{10} dt$. We obtain

$$\int \frac{1}{10} e^{-t/10} dt = - \int e^u du = -e^u + C = -e^{-t/10} + C$$

So the desired probability is

$$\int_{10}^{\infty} \frac{1}{10} e^{-t/10} dt = -e^{-t/10} \Big|_{t=10}^{\infty} = \left(\lim_{t \rightarrow \infty} -e^{-t/10} \right) - \left(-e^{-10/10} \right) = e^{-1} = 1/e$$

So the correct answer is **A**, namely, $1/e$.