

Math 104
Midterm Exam 3
April 11, 2007

Name _____

Circle exactly one professor/TA combination:

Professor:	Dr. Ackerman	Dr. Popa	Dr. van Erp	Dr. Ward	Dr. Ward
TA:	Andrew Rupinski	Jen Hom	Andrew Bressler	Alexa Mater	Tim DeVries
time:	1 PM, MWF	10:30 AM, TR	3 PM, TR	10 AM, MWF	10 AM, MWF

Write all answers (A, B, C, D, E, F) in the spaces provided below!

- | | |
|----------|-----------|
| 1. _____ | 6. _____ |
| 2. _____ | 7. _____ |
| 3. _____ | 8. _____ |
| 4. _____ | 9. _____ |
| 5. _____ | 10. _____ |

Score: _____ (100 points possible)

1. The testing booklet contains 10 questions.
2. No calculators are permitted.
3. One piece of paper (8.5 in. by 11 in.) is permitted, with writing on both sides allowed.
4. There is no penalty for guessing.
5. No partial credit will be given.
6. Write all calculations on the pages provided. Extra pages are available if needed.

1. The curve

$$r = \cos(5\theta)$$

is a calculus flower. Find the area of the region contained in one leaf.

A.) $\pi/40$ B.) $\pi/20$ C.) $\pi/10$ D.) $\pi/5$ E.) $\pi/2$ F.) π

Answer. One leaf is generated for $-\pi/10 \leq \theta \leq \pi/10$. (Another leaf is generated, for instance, for $\pi/10 \leq \theta \leq 3\pi/10$.) So the area of one leaf is

$$\begin{aligned} A &= \int_{-\pi/10}^{\pi/10} \frac{1}{2} (\cos(5\theta))^2 d\theta \\ &= \int_{-\pi/10}^{\pi/10} \frac{1}{2} \cdot \frac{1}{2} (1 + \cos(10\theta)) d\theta \\ &= \frac{1}{4} \left(\theta + \frac{\sin(10\theta)}{10} \right) \Big|_{\theta=-\pi/10}^{\pi/10} \\ &= \frac{1}{4} \left[\left(\frac{\pi}{10} + \frac{\sin(\pi)}{10} \right) - \left(-\frac{\pi}{10} + \frac{\sin(-\pi)}{10} \right) \right] \\ &= \frac{1}{4} \left(\frac{\pi}{5} \right) \\ &= \pi/20 \end{aligned}$$

So the correct answer is **B**, namely, $\pi/20$.

2. Find a polar equation for the curve represented by the Cartesian equation

$$x^2 + y^2 = 2x.$$

A.) $r = 2 \sin \theta$

B.) $r = 2 \cos \theta$

C.) $r = \sin(2\theta)$

D.) $r = \cos(2\theta)$

E.) $r = \sin \theta + \cos \theta$

F.) $r = \frac{\sin(2\theta)}{2}$

Answer. We recall $x = r \cos \theta$ and $y = r \sin \theta$, so the desired equation is

$$r^2 \cos^2 \theta + r^2 \sin^2 \theta = 2r \cos \theta.$$

Since $\cos^2 \theta + \sin^2 \theta = 1$, the equation above simplifies to $r^2 = 2r \cos \theta$, or more simply, $r = 2 \cos \theta$.

So the correct answer is **B**, namely, $r = 2 \cos \theta$.

3. Find the length of the curve.

$$x = \frac{1}{2}t^2, \quad y = \frac{4}{3}(t+1)^{3/2}, \quad 0 \leq t \leq 2$$

A.) $\frac{56}{3}$

B.) 2

C.) 4

D.) 6

E.) 8

F.) 10

Answer. The length of the curve is

$$\begin{aligned} L &= \int_0^2 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\ &= \int_0^2 \sqrt{\left(\frac{1}{2}(2t)\right)^2 + \left(\frac{4}{3} \cdot \frac{3}{2}(t+1)^{1/2}\right)^2} dt \\ &= \int_0^2 \sqrt{(t)^2 + (2(t+1)^{1/2})^2} dt \\ &= \int_0^2 \sqrt{t^2 + 4t + 4} dt \\ &= \int_0^2 \sqrt{(t+2)^2} dt \\ &= \int_0^2 (t+2) dt \\ &= \left(\frac{t^2}{2} + 2t\right) \Big|_{t=0}^2 \\ &= 6 \end{aligned}$$

So the correct answer is **D**, namely, 6.

4. Find the area of the surface obtained by rotating the curve

$$x = \cos(t^2), \quad y = \sin(t^2), \quad 0 \leq t \leq \sqrt{\pi},$$

about the x -axis.

- A.) 0 B.) π C.) 2π D.) 3π E.) 4π F.) 5π

Answer. The area of the surface is

$$\begin{aligned} S &= \int_0^{\sqrt{\pi}} 2\pi \sin(t^2) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\ &= \int_0^{\sqrt{\pi}} 2\pi \sin(t^2) \sqrt{(-\sin(t^2) \cdot 2t)^2 + (\cos(t^2) \cdot 2t)^2} dt \\ &= \int_0^{\sqrt{\pi}} 2\pi \sin(t^2) \sqrt{(4t^2)(\sin^2(t^2) + \cos^2(t^2))} dt \\ &= \int_0^{\sqrt{\pi}} 2\pi \sin(t^2) \sqrt{4t^2} dt \\ &= \int_0^{\sqrt{\pi}} 2\pi \sin(t^2) \cdot 2t dt \end{aligned}$$

We use $u = t^2$ and $du = 2t dt$, so

$$\int 2\pi \sin(t^2) \cdot 2t dt = \int 2\pi \sin u du = -2\pi \cos u + C = -2\pi \cos(t^2) + C$$

Thus

$$S = (-2\pi \cos(t^2)) \Big|_{t=0}^{\sqrt{\pi}} = -2\pi(\cos(\pi) - \cos(0)) = -2\pi(-1 - 1) = 4\pi$$

So the correct answer is **E**, namely, 4π .

5. Consider the following two series:

$$\sum_{n=1}^{\infty} \frac{\cos(1/n)}{n^2} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{\cos(2n)}{n^2}$$

Which of the following is true?

- A.) both series are absolutely convergent
- B.) both series are conditionally convergent
- C.) both series are divergent
- D.) one series is absolutely convergent, and one series is conditionally convergent
- E.) one series is absolutely convergent, and one series is divergent
- F.) one series is conditionally convergent, and one series is divergent

Answer. Consider the sum of the absolute-value version of the first series:

$$\sum_{n=1}^{\infty} \left| \frac{\cos(1/n)}{n^2} \right| \leq \sum_{n=1}^{\infty} \frac{1}{n^2}$$

We know that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent (this is a p -series with $p = 2$). So $\sum_{n=1}^{\infty} \left| \frac{\cos(1/n)}{n^2} \right|$ is convergent too (by the Comparison Test). So $\sum_{n=1}^{\infty} \frac{\cos(1/n)}{n^2}$ is an absolutely convergent series.

Now consider the sum of the absolute-value version of the second series:

$$\sum_{n=1}^{\infty} \left| \frac{\cos(2n)}{n^2} \right| \leq \sum_{n=1}^{\infty} \frac{1}{n^2}$$

We know that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent (this is a p -series with $p = 2$). So $\sum_{n=1}^{\infty} \left| \frac{\cos(2n)}{n^2} \right|$ is convergent too (by the Comparison Test). So $\sum_{n=1}^{\infty} \frac{\cos(2n)}{n^2}$ is an absolutely convergent series.

So the correct answer is **A**, namely, both series are absolutely convergent.

6. Determine whether the series

$$\sum_{n=0}^{\infty} \frac{2^n + 5^n}{10^n}$$

is convergent or divergent. If it is convergent, find its sum.

- A.) the series is convergent and sums to $7/10$
- B.) the series is convergent and sums to $3/4$
- C.) the series is convergent and sums to $5/4$
- D.) the series is convergent and sums to $13/4$
- E.) the series is convergent and sums to 7
- F.) the series is divergent

Answer. We first note that

$$\sum_{n=0}^{\infty} \frac{2^n + 5^n}{10^n} = \sum_{n=0}^{\infty} \left(\frac{2^n}{10^n} + \frac{5^n}{10^n} \right) = \sum_{n=0}^{\infty} \left(\left(\frac{2}{10} \right)^n + \left(\frac{5}{10} \right)^n \right) = \sum_{n=0}^{\infty} \left(\left(\frac{1}{5} \right)^n + \left(\frac{1}{2} \right)^n \right)$$

We know that $\sum_{n=0}^{\infty} \left(\frac{1}{5} \right)^n = \frac{1}{1-\frac{1}{5}} = \frac{1}{4/5} = \frac{5}{4}$ by using the formula $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for $-1 < x < 1$ (in this case, $x = 1/5$).

Also, we know that $\sum_{n=0}^{\infty} \left(\frac{1}{2} \right)^n = \frac{1}{1-\frac{1}{2}} = \frac{1}{1/2} = 2$ by again using the formula $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for $-1 < x < 1$ (in this case, $x = 1/2$).

So the original series is

$$\sum_{n=0}^{\infty} \frac{2^n + 5^n}{10^n} = \sum_{n=0}^{\infty} \left(\left(\frac{1}{5} \right)^n + \left(\frac{1}{2} \right)^n \right) = \frac{5}{4} + 2 = \frac{13}{4}$$

So the correct answer is **D**, namely, the series is convergent and sums to $13/4$.

7. Consider the following three series:

$$\sum_{n=1}^{\infty} \frac{n}{1+n^2} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{\tan^{-1} n}{1+n^2} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{\sqrt{n^4+1}}{1+n^2}$$

Which of the following is true?

- A.) only the first series converges
- B.) only the second series converges
- C.) only the third series converges
- D.) both the first and second series converge
- E.) both the first and third series converge
- F.) both the second and third series converge

Answer. The first series diverges. To see this, note that all terms $\frac{n}{1+n^2}$ are positive and are approximately $\frac{n}{n^2}$ (in other words, $1/n$) when n is large. So use the Limit Comparison Test with $a_n = \frac{n}{1+n^2}$ for the original series and $b_n = \frac{n}{n^2}$ for the series to compare. We have

$$\frac{a_n}{b_n} = \frac{\frac{n}{1+n^2}}{\frac{n}{n^2}} = \frac{n^2}{1+n^2} = \frac{1}{\frac{1}{n^2} + 1} \quad \text{so} \quad \lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 1 = c > 0$$

So by the Limit Comparison Test, $\sum_{n=1}^{\infty} \frac{n}{1+n^2}$ and $\sum_{n=1}^{\infty} \frac{n}{n^2}$ either both converge or both diverge. We notice that $\sum_{n=1}^{\infty} \frac{n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n}$ is the harmonic series, which diverges, so the original series diverges too.

The second series converges. Here are two quick ways to see this:

Method 1. Use the Integral Test, comparing to $\int_1^{\infty} \frac{\tan^{-1} x}{1+x^2} dx$. Write $u = \tan^{-1} x$ and $du = \frac{1}{1+x^2} dx$, so $\int \frac{\tan^{-1} x}{1+x^2} dx = \int u du = \frac{1}{2}u^2 + C = \frac{1}{2}(\tan^{-1} x)^2 + C$, and thus $\int_1^{\infty} \frac{\tan^{-1} x}{1+x^2} dx = \frac{1}{2}((\pi/2)^2 - (\pi/4)^2)$, which is finite. So the series $\sum_{n=1}^{\infty} \frac{\tan^{-1} n}{1+n^2}$ converges by the Integral Test.

Method 2. Use the Comparison Test, since all terms of $\sum_{n=1}^{\infty} \frac{\tan^{-1} n}{1+n^2}$ are positive, with $\frac{\tan^{-1} n}{1+n^2} \leq \frac{\pi/2}{1+n^2} \leq \frac{\pi/2}{n^2}$. So $\sum_{n=1}^{\infty} \frac{\tan^{-1} n}{1+n^2} \leq \sum_{n=1}^{\infty} \frac{\pi/2}{n^2} = \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{1}{n^2}$, which is a convergent p -series (with $p = 2$). So $\sum_{n=1}^{\infty} \frac{\tan^{-1} n}{1+n^2}$ converges too.

The third series diverges. To see this, note that the terms of the series do not converge to 0. In fact, the terms of the series converge to 1. We compute

$$\frac{\sqrt{n^4+1}}{1+n^2} = \frac{(\frac{1}{n^2})\sqrt{n^4+1}}{\frac{1}{n^2}+1} = \frac{\sqrt{(\frac{1}{n^4})(n^4+1)}}{\frac{1}{n^2}+1} = \frac{\sqrt{1+\frac{1}{n^4}}}{\frac{1}{n^2}+1}$$

so $\lim_{n \rightarrow \infty} \frac{\sqrt{n^4+1}}{1+n^2} = \frac{\sqrt{1+0}}{0+1} = 1$. Since the terms of the series $\sum_{n=1}^{\infty} \frac{\sqrt{n^4+1}}{1+n^2}$ converge to 1 (but NOT to 0), then the series diverges.

So the correct answer is **B**, namely, only the second series converges.

8. Consider the following two sequences:

$$a_n = \frac{\sin(n)}{n^2 + 1} \quad \text{and} \quad a_n = (-1)^n \cdot \frac{n + 2}{n^2 + 1}$$

Which of the following is true?

- A.) both of the sequences converge to 0
- B.) both of the sequences converge to 1
- C.) both of the sequences diverge
- D.) one of the sequences converges to 0, and one of the sequences converges to 1
- E.) one of the sequences converges to 0, and one of the sequences diverges
- F.) one of the sequences converges to 1, and one of the sequences diverges

Answer. The first series converges to 0. To see this, use the squeeze theorem. We have

$$\frac{-1}{n^2 + 1} \leq \frac{\sin(n)}{n^2 + 1} \leq \frac{1}{n^2 + 1}$$

Since $\frac{-1}{n^2+1} \rightarrow 0$ and $\frac{1}{n^2+1} \rightarrow 0$, then by the Squeeze Theorem, $\frac{\sin(n)}{n^2+1} \rightarrow 0$ too.

The second series converges to 0. To see this, again use the squeeze theorem. We have

$$\frac{(-1)(n + 2)}{n^2 + 1} \leq \frac{(-1)^n(n + 2)}{n^2 + 1} \leq \frac{n + 2}{n^2 + 1}$$

Since $\frac{(-1)(n+2)}{n^2+1} = -\frac{\frac{1}{n} + \frac{2}{n^2}}{1 + \frac{1}{n^2}} \rightarrow -\frac{0+0}{1+0} = 0$ and $\frac{n+2}{n^2+1} = \frac{\frac{1}{n} + \frac{2}{n^2}}{1 + \frac{1}{n^2}} \rightarrow \frac{0+0}{1+0} = 0$, then by the Squeeze Theorem, $\frac{(-1)^n(n+2)}{n^2+1} \rightarrow 0$ too.

So the correct answer is **A**, namely, both of the sequences converge to 0.

9. Determine whether the series is convergent or divergent. If the series is convergent, find its sum.

$$9 - 6 + 4 - \frac{8}{3} + \frac{16}{9} - \frac{32}{27} + \cdots$$

- A.) the series is convergent and sums to $2/5$
- B.) the series is convergent and sums to $3/5$
- C.) the series is convergent and sums to 3
- D.) the series is convergent and sums to $18/5$
- E.) the series is convergent and sums to $27/5$
- F.) the series is divergent

Answer. We write the terms as

$$\begin{aligned} a_0 &= 9 \\ a_1 &= -6 \\ a_2 &= 4 \\ a_3 &= -8/3 \\ a_4 &= 16/9 \\ a_5 &= -32/27 \end{aligned}$$

Using $(-1)^n$ on each term will allow the sign of a_n to be correct. Using 2^n in the numerator of each a_n will handle the power of 2 correctly in each a_n . Finally, using 3^{n-2} in the denominator of each a_n will handle the power of 3 correctly in each a_n . So $a_n = \frac{(-1)^n 2^n}{3^{n-2}}$. The sum becomes

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{3^{n-2}} &= \sum_{n=0}^{\infty} \frac{3^2 (-1)^n 2^n}{3^2 \cdot 3^{n-2}} \\ &= 9 \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{3^n} \\ &= 9 \sum_{n=0}^{\infty} \left(-\frac{2}{3}\right)^n \\ &= 9 \left(\frac{1}{1 - (-\frac{2}{3})}\right) \\ &= 9 \left(\frac{1}{5/3}\right) \\ &= 9 \left(\frac{3}{5}\right) \\ &= 27/5 \end{aligned}$$

So the correct answer is **E**, namely, the series is convergent and sums to $27/5$.

10. Consider the following two series:

$$\sum_{n=1}^{\infty} \frac{(-1)^n n^3}{5^n} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{n^3}{n!}$$

Which of the following is true?

- A.) the first series converges absolutely, and the second series converges
- B.) the first series converges absolutely, and the second series diverges
- C.) the first series converges conditionally, and the second series converges
- D.) the first series converges conditionally, and the second series diverges
- E.) the first series diverges, and the second series converges
- F.) the first series diverges, and the second series diverges

Answer. The first series converges absolutely. To see this, write the sum of the absolute-values of the terms:

$$\sum_{n=1}^{\infty} \frac{n^3}{5^n}$$

and use the Ratio Test, with $a_n = \frac{n^3}{5^n}$ and $a_{n+1} = \frac{(n+1)^3}{5^{n+1}}$. So

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^3 5^n}{n^3 5^{n+1}} = \frac{1}{5} \left(\frac{n+1}{n} \right)^3 = \frac{1}{5} \left(1 + \frac{1}{n} \right)^3$$

So $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \frac{1}{5} = L < 1$, so the series $\sum_{n=1}^{\infty} \frac{n^3}{5^n}$ converges by the Ratio Test. Thus the series $\sum_{n=1}^{\infty} \frac{(-1)^n n^3}{5^n}$ converges absolutely.

The second series converges. To see this, use the Ratio Test, with $a_n = \frac{n^3}{n!}$ and $a_{n+1} = \frac{(n+1)^3}{(n+1)!}$. So

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^3 n!}{n^3 (n+1)!} = \left(\frac{n+1}{n} \right)^3 \frac{n!}{(n+1)!} = \left(1 + \frac{1}{n} \right)^3 \frac{1}{n+1}$$

So $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = (1)(0) = 0 = L < 1$, so the series converges by the Ratio Test.

So the correct answer is **A**, namely, the first series converges absolutely, and the second series converges.