

## 1. Conditions of integrability

Consider the function  $f: [-1, 1] \rightarrow \mathbb{R}$  defined by  $f(x) = -1$  for  $x \in [-1, 0]$  and  $f(x) = 1$  for  $x \in (0, 1]$ .

**Exercise 1.** Prove that this function is Riemann integrable by using Riemann's condition. For every  $\varepsilon > 0$  describe an explicit partition  $P$  of the interval  $[-1, 1]$  into *three* subintervals,

$$-1 = x_0 < x_1 < x_2 < x_3 = 1,$$

such that the  $U_P - L_P < \varepsilon$ .

**Exercise 2\*.** Find an explicit value for  $\delta > 0$ , such that for *every* partition

$$-1 = x_0 < x_1 < x_2 < \cdots < x_N = 1,$$

that has  $x_{i+1} - x_i < \delta$  for all  $i = 0, 1, \dots, N - 1$ , and with arbitrary choice of tags  $x_i \leq t_i \leq x_{i+1}$ , the corresponding Riemann sum

$$\sum_{i=0}^{N-1} f(t_i)(x_{i+1} - x_i)$$

differs no more than  $\pm \frac{1}{100}$  from the exact value of the integral  $\int_{-1}^1 f(x) dx$ .

Comment: You don't have to give an abstract proof, but rather an *explicit* value for  $\delta$  for this specific function  $f$ , and an explanation how you obtained it. Study the proof of Darboux's Theorem 8.1.2 (page 474ff), and use the result of Exercise 1. You don't need to know the value of the integral to do Exercise 2.

## 2. Alternative definition of the Riemann integral

**Definition 1.** A *directed set*  $\Omega$  is a set that has an inequality between its elements, denoted  $\omega_1 \geq \omega_2$ , for which the following properties hold,

- (1)  $\omega \geq \omega$  for every  $\omega \in \Omega$ ,
- (2) whenever  $\omega_1 \geq \omega_2$  and  $\omega_2 \geq \omega_3$ , then also  $\omega_1 \geq \omega_3$ ,
- (3) for any two elements  $\omega_1, \omega_2 \in \Omega$ , there exists at least one element  $\omega_3 \in \Omega$  such that  $\omega_3 \geq \omega_1$  and  $\omega_3 \geq \omega_2$ .

Properties (1) and (2) are properties of a partial inequality. The inequality is "partial" because it is not required that *every* pair of elements  $\omega_1, \omega_2$  can be compared: it is possible that neither  $\omega_1 \geq \omega_2$  nor  $\omega_2 \geq \omega_1$ . Property (3) makes the inequality "directed".

**Exercise 3.** Show that the natural numbers with the usual inequality ( $7 \geq 4$ , etc.) form a directed set.

**Exercise 4.** Let  $\Omega = \mathbb{N} \times \mathbb{N}$  be the set of ordered pairs of natural numbers  $\omega = (m, n)$ , with  $(m_1, n_1) \geq (m_2, n_2)$  if and only if both  $m_1 \geq m_2$  and  $n_1 \geq n_2$ . Show that  $\Omega$  is a directed set.

Recall that a sequence is a collection of numbers  $a_n$  that are indexed by an integer  $n \in \mathbb{N}$ . A sequence  $a_n$  converges to a limit  $L$ ,

$$\lim_{n \rightarrow \infty} a_n = L,$$

if for every  $\epsilon > 0$  there exists an integer  $N$  such that

$$|a_n - L| < \epsilon$$

for every  $n \geq N$ .

Replacing the index set  $\mathbb{N}$  by an arbitrary directed sets  $\Omega$  we obtain a useful generalization of sequences and limits.

**Definition 2.** A *net* is a collection of numbers  $a_\omega$  that are indexed by elements of a directed set  $\Omega$ .

Formally, a net is a simply a *function*  $a: \Omega \rightarrow \mathbb{R}$ , i.e., there is a unique real number  $a_\omega$  assigned to every element  $\omega \in \Omega$  in the directed set.

**Definition 3.** A net  $a_\omega$  converges to a limit  $L$ ,

$$\lim_{\omega \rightarrow \infty} a_\omega = L$$

if for every  $\epsilon > 0$  there exists an index  $\omega_0 \in \Omega$  such that

$$|a_\omega - L| < \epsilon$$

for every  $\omega \geq \omega_0$ .

**Exercise 5.** Let  $\Omega$  be as in Exercise 2. Define a net  $a_\omega$  by the formula

$$a_\omega = a_{(m,n)} = \frac{(-1)^{m+n}}{\min(m,n)}.$$

Here  $\min(m,n)$  denotes the minimum value of  $\{m,n\}$ . Show that the iterated limits

$$\lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} a_{(m,n)}, \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} a_{(m,n)}$$

do not converge. By contrast, prove that the net  $a_\omega$  does converge, with limit

$$\lim_{\omega \rightarrow \infty} a_\omega = 0$$

Now let  $f: A \subset \mathbb{R}^n \rightarrow \mathbb{R}$  be a real-valued function defined on a bounded domain  $A$ . Assume that  $A$  is enclosed in a rectangular box  $B \subseteq A$ .

**Exercise 6.** Let  $\Omega$  be the collection of all tagged partitions  $\omega = (P, t)$  of  $B$ . We define  $(P, t) \geq (P', t')$  to mean that  $P$  is a refinement of  $P'$ . Observe that the choice of tags does not play a role in determining the inequality  $\geq$ . Prove that  $\Omega$  is a directed set.

With this definition of the directed set of tagged partitions, the Riemann sums

$$I_\omega = \sum_{S \in P} f(t_S) \nu(S)$$

form a net with index  $\omega = (P, t) \in \Omega$ .

**Definition 4.** A function is Riemann integrable if the net of its Riemann sums converges. In that case the limit

$$\lim_{\omega \rightarrow \infty} I_\omega = I$$

is the Riemann integral.

**Exercise 7\*.** Prove that this definition of Riemann integrability is equivalent to the standard definition, by using Riemann's condition and Darboux's Theorem.