1. According to Thm 8.2.4, for a countable family of sets $A_1, A_2, A_3, \cdots$, in which each set A_i has measure zero, the union

   $$A_1 \cup A_2 \cup A_3 \cup \cdots$$

   has also measure zero.

   a) Prove that the union of every *finite* family of zero *volume* sets $A_1, A_2, \cdots, A_N$, also has zero volume.

   b) Give an example of a countable family of sets with zero volume where the union does *not* have zero volume.

2. (Exercise 8.2.1) Prove that the circle

   $$A = \{(x, y) \in \mathbb{R}^2 \mid \ x^2 + y^2 = 1\}$$

   has zero volume in $\mathbb{R}^2$.

   Hint: Construct a regular polygon with N sides, whose vertices are points on the circle. On each side of this polygon, construct a rectangle that is big enough to cover the arc of the circle between the two vertices. Express the size of the base and height of the rectangle in a formula involving N (using trigonometry), and then derive a formula for the total volume of the rectangles. Then see what happens as $N \to \infty$.

   Explain why this proves that the circle has volume zero.

3. (Exercise 5, p.490) In $\mathbb{R}^3$, prove that any subset of the xy plane has measure zero.

4. (Exercise 11b, p.491)

   a) Easy part: Prove that if a set A has zero volume, than it also has measure zero.

   b) Harder part: If A is a *compact* set that has measure zero, prove that it also has zero volume.

5. a) (Exercise 24, p.492) Give an example to show that the following is not equivalent to the integrability of f, i.e., construct an explicit function f that is *not* integrable, for which the following statement is true.

   For any $\varepsilon > 0$, there is a $\delta > 0$ such that if P is any partition into rectangles $S_1, \ldots, S_N$ with sides $< \delta$, there exist tags $t_1 \in S_1, \ldots, t_N \in S_N$ such that

   $$I - \varepsilon < I_P < I + \varepsilon,$$

   where I_P is the Riemann sum

   $$I_P = \sum_{i=1}^{N} f(t_i)\nu(S_i).$$

   b) What is the difference between this (false) condition and the condition in Darboux's Theorem?

6. (Exercise 38, p. 494) Let the function $f\colon [0,1] \to \mathbb{R}$ be defined by $f(x) = 0$ if x is irrational and $f(x) = 1/q$ if x is a rational number equal to p/q for integers p and q with no common factor.

   Prove that f is integrable, with $\int_0^1 f(x)dx = 0$.

   Hint: Instead of using Lebesgue's Theorem, it is perhaps easier in this case to construct, for every $\varepsilon > 0$, an explicit partition P for which the upper sum $U_P < \varepsilon$. Use this to prove that $\sup L_P = \inf U_P = 0$.