

Homework 4

Math 361, Fall 2007

1. Let E be a subset of \mathbb{R}^n , not necessarily measurable. Show that

$$\begin{aligned}\mu^*(E) &= \inf \{ \mu(A) \mid E \subseteq A, A \text{ measurable} \} \\ \mu_*(E) &= \sup \{ \mu(A) \mid A \subseteq E, A \text{ measurable} \}\end{aligned}$$

2. Let

$$A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$$

be an increasing sequence of measurable subsets of \mathbb{R}^n , with union $A = \bigcup^\infty A_i$. Show that

$$\mu(A) = \lim_{i \rightarrow \infty} \mu(A_i).$$

Prove that this is also true if

$$A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$$

is a decreasing sequence, and $A = \bigcap^\infty A_i$. (You may assume that $\mu(A_1) < \infty$.)

3. Let Σ be the collection of all *countable* subsets in \mathbb{R} . For a set $E \in \Sigma$, let $\mu(E)$ be the number of *rational* points contained in E (possible ∞). Show that Σ is a σ -algebra in \mathbb{R} , and that μ is a measure on Σ .
4. Let A be a subset of \mathbb{R}^n , not necessarily measurable. Assume that A is contained in the cube $B = [0, 1] \times \cdots \times [0, 1]$.

Let 1_A denote the characteristic function of A , i.e., $1_A(x) = 1$ if $x \in A$, and $1_A(x) = 0$ if $x \notin A$. If P denotes a division of B into disjoint measurable subsets, as in the definition of Lebesgue integral, show that for the function 1_A

$$\begin{aligned}\sup_P L_P &= \mu_*(A) \\ \inf_P U_P &= \mu^*(A)\end{aligned}$$

What can you conclude about the existence of the Lebesgue integral $\int 1_A$?