Homework 8 *Math 361, Fall 2007*

1. Let M be a linear subspace of a Hilbert space H.

   (a) Prove that the orthogonal complement $M^\perp$ is a linear subspace.

   (b) Prove that $M^\perp$ is always a *closed* subspace, even if M is not.

   (Use this definition: a subset $X \subseteq Y$ in a metric space Y is closed if every Cauchy sequence of points in X converges to a point in X. You also need to use the Cauchy-Schwartz inequality.)

2. Consider the Hilbert space $H = L^2([-1,1])$ with inner product

$$\langle f, g \rangle = \int_{-1}^{1} f\bar{g}.$$

   A function $f \in H$ is called *even* if $f(-x) = f(x)$ for almost all x, and *odd* if $f(-x) = -f(x)$ for almost all x.

   Prove that the every even function f is orthogonal to every odd function g (for $f, g \in H$).

   Prove that every function $f \in H$ is the sum $f = f_0 + f_1$ of an even function $f_0 \in H$ and an odd function $f_1 \in H$.

   Prove that the orthogonal complement of the space of even functions in H is the space of odd functions, and vice versa.

   Prove that the collections of even and odd functions are closed subspaces of H.

3. In the Hilbert space $H = L^2([-\pi,\pi])$ with inner product

$$\langle f, g \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f\bar{g}$$

   the collection of functions

$$s_n(x) = \sin(nx), \qquad n = 1, 2, 3, \ldots$$
$$c_n(x) = \cos(nx), \qquad n = 1, 2, 3, \ldots$$
$$c_0(x) = \frac{1}{\sqrt{2}},$$

   forms a complete orthonormal system. Every $f \in L^2$ has a corresponding Fourier series

$$f = A_0 + \sum_{i=1}^{\infty} \left(A_n \cos(nx) + B_n \sin(nx)\right).$$

   If f_0 denotes the even part of f, and f_1 the odd part (see previous exercise), then prove that

$$f_0 = A_0 + \sum_{i=1}^{\infty} A_n \cos(nx)$$
$$f_1 = \sum_{i=1}^{\infty} B_n \sin(nx).$$

4. Do exercise 4 (a, b, c), page 560 in the text book.