Math 104
Answers for Midterm Exam 1
February 7, 2007

1. Find the value of c so that the line $y = c$ divides the region bounded by the curves

$$y = x^2 \qquad \text{and} \qquad y = 4$$

into two regions of equal area.

A.) $1/2$ B.) 1 C.) $2^{1/3}$ D.) $2^{2/3}$ E.) 2 F.) $2^{4/3}$

Answer. We want to find the value of c so that

$$\int_{-2}^{2} (4 - x^2)\, dx = 2 \int_{-\sqrt{c}}^{\sqrt{c}} (c - x^2)\, dx$$

The left-hand-side is

$$\left(4x - \frac{x^3}{3}\right)\Bigg|_{x=-2}^{2} = \left(8 - \frac{8}{3}\right) - \left(-8 - \frac{-8}{3}\right) = 32/3$$

The right-hand-side is

$$2\left(cx - \frac{x^3}{3}\right)\Bigg|_{x=-\sqrt{c}}^{\sqrt{c}} = 2\left[\left(c^{3/2} - \frac{c^{3/2}}{3}\right) - \left(-c^{3/2} - \frac{-c^{3/2}}{3}\right)\right] = \frac{8}{3}c^{3/2}$$

So we conclude that $\frac{32}{3} = \frac{8}{3}c^{3/2}$, or in other words, $4 = c^{3/2}$, so $c = 4^{2/3} = (2^2)^{2/3} = 2^{4/3}$.

So the correct answer is **F**, namely, $2^{4/3}$.

2. Find the volume of the solid of revolution obtained by rotating the region enclosed by the graphs of

$$y = x^{1/3}, \qquad y = 0, \qquad x = 1,$$

about the line $x = 2$.

A.) $3\pi/14$ B.) $9\pi/14$ C.) $\pi/7$ D.) $5\pi/7$ E.) $6\pi/7$ F.) $15\pi/7$

Answer.

Method 1. Using shells (soup labels), we see that each thin shell (soup label) has height $x^{1/3}$, radius $2 - x$, and thinness dx. So each shell (soup label) has volume

$$2\pi(2 - x)x^{1/3}\,dx$$

So the total volume is

$$\int_0^1 2\pi(2 - x)x^{1/3}\,dx = 2\pi \int_0^1 (2x^{1/3} - x^{4/3})\,dx = 2\pi \left(2\frac{x^{4/3}}{4/3} - \frac{x^{7/3}}{7/3} \right)\Bigg|_{x=0}^{1} = 15\pi/7$$

Method 2. Using washers, we see that each thin washer has outer radius $2 - x = 2 - y^3$, inner radius $2 - 1 = 1$, and thinness dy. So each washer has volume

$$\pi((2 - y^3)^2 - 1^2)\,dy$$

So the total volume is

$$\int_0^1 \pi((2 - y^3)^2 - 1^2)\,dy = \pi \int_0^1 (3 - 4y^3 + y^6)\,dy = \pi \left(3y - 4\frac{y^4}{4} + \frac{y^7}{7} \right)\Bigg|_{y=0}^{1} = 15\pi/7$$

So the correct answer is **F**, namely, $15\pi/7$.

3. Find the volume of the solid of revolution obtained by rotating the region enclosed by the graphs of

$$y = \sqrt{x^3}\,, \qquad y = 1\,, \qquad x = 0\,,$$

about the line $y = 1$.

A.) $5\pi/16$ B.) $6\pi/17$ C.) $7\pi/18$ D.) $8\pi/19$ E.) $9\pi/20$ F.) $10\pi/21$

Answer.

Method 1. Using shells (soup labels), we see that each thin shell (soup label) has height $x = y^{2/3}$, radius $1 - y$, and thinness dy. So each shell (soup label) has volume

$$(2\pi)(1-y)(y^{2/3})\,dy$$

So the total volume is

$$\int_0^1 2\pi(1-y)(y^{2/3})\,dy = 2\pi \int_0^1 (y^{2/3} - y^{5/3})\,dy = 2\pi \left(\frac{y^{5/3}}{5/3} - \frac{y^{8/3}}{8/3} \right) \Bigg|_{y=0}^{1} = 9\pi/20$$

Method 2. Using discs, we see that each thin disc has radius $1 - x^{3/2}$ and thinness dx. So each disc has volume

$$\pi(1 - x^{3/2})^2\,dx$$

So the total volume is

$$\int_0^1 \pi(1 - x^{3/2})^2\,dx = \pi \int_0^1 (1 - 2x^{3/2} + x^3)\,dx = \pi \left(x - 2\frac{x^{5/2}}{5/2} + \frac{x^4}{4} \right) \Bigg|_{x=0}^{1} = 9\pi/20$$

So the correct answer is **E**, namely, $9\pi/20$.

4. Find the volume of the solid of revolution obtained by rotating the region enclosed by the graphs of

$$y = x - 2 \qquad \text{and} \qquad x = 4 - y^2$$

about the y-axis.

A.) $27\pi/5$ B.) $54\pi/5$ C.) $81\pi/5$ D.) $108\pi/5$ E.) $135\pi/5$ F.) $162\pi/5$

Answer. We first find the points of intersection of the two graphs. The two graphs intersect when $y+2 = 4-y^2$, i.e., when $y^2+y-2 = 0$, i.e., when $(y+2)(y-1) = 0$, namely, when $y = -2$ and also when $y = 1$.

Washers are easier to use in this problem. (With shells (soup labels), we would need to do two integrations.) Using washers, we see that each thin washer has outer radius $4 - y^2$, inner radius $y + 2$, and thinness dy. So each washer has volume

$$\pi((4 - y^2)^2 - (y + 2)^2)\,dy$$

So the total volume is

$$\begin{aligned}\int_{-2}^{1} \pi((4 - y^2)^2 - (y + 2)^2)\,dy &= \pi \int_{-2}^{1} ((16 - 8y^2 + y^4) - (y^2 + 4y + 4))\,dy \\ &= \pi \int_{-2}^{1} (y^4 - 9y^2 - 4y + 12)\,dy \\ &= \pi \left(\frac{y^5}{5} - 9\frac{y^3}{3} - 4\frac{y^2}{2} + 12y \right)\Bigg|_{y=-2}^{1} \\ &= 108\pi/5\end{aligned}$$

So the correct answer is **D**, namely, $108\pi/5$.

5. Consider

$$f(x) = e^{2\tan x}, \qquad \text{restricted to the interval } x \in \left(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right).$$

Find $(f^{-1})'(1)$.

A.) 0 B.) 1/2 C.) $\pi/4$ D.) 1 E.) 2 F.) e^2

Answer. For convenience, we write $g(x) = f^{-1}(x)$ for the inverse of $f(x)$. We have

$$(f^{-1})'(1) = g'(1) = \frac{1}{f'(g(1))}$$

We know that $f'(x) = (e^{2\tan x})(2\sec^2 x)$. Now we find $g(1)$. To do this, we need the value of x so that $f(x) = 1$, i.e., $e^{2\tan x} = 1$, i.e., $2\tan x = 0$, i.e., $x = 0$. Since $f(0) = 1$, then $g(1) = 0$.

We conclude that

$$(f^{-1})'(1) = \frac{1}{f'(0)} = \frac{1}{(e^{2\tan 0})(2\sec^2(0))} = \frac{1}{(1)(2)} = 1/2$$

So the correct answer is **B**, namely, 1/2.

6. Integrate:

$$\int_0^{\frac{1}{2}\ln(3)} \frac{e^x}{1+e^{2x}}\,dx$$

A.) $\pi/3$ B.) $\frac{\sqrt{3}}{4(1+\sqrt{3})^2}$ C.) $\frac{\sqrt{3}}{3}-1$ D.) $\arcsin(\sqrt{3})-\frac{\pi}{2}$ E.) $\pi/12$ F.) $\sqrt{3}-1$

Answer. In the denominator, we notice that $e^{2x}=(e^x)^2$. So we first use u-substitution to make the problem easier. We write $u=e^x$ and $du=e^x\,dx$. So

$$\int \frac{e^x}{1+e^{2x}}\,dx = \frac{1}{1+u^2}\,du = \tan^{-1}(u)+C = \tan^{-1}(e^x)+C$$

So

$$\int_0^{\frac{1}{2}\ln(3)} \frac{e^x}{1+e^{2x}}\,dx = \tan^{-1}(e^x)\Big|_{x=0}^{\frac{1}{2}\ln(3)}$$

We notice that $\frac{1}{2}\ln(3)=\ln(\sqrt{3})$, so $e^{(\frac{1}{2}\ln(3))}=e^{\ln(\sqrt{3})}=\sqrt{3}$. Also $e^0=1$. So the line above simplifies to

$$\int_0^{\frac{1}{2}\ln(3)} \frac{e^x}{1+e^{2x}}\,dx = \tan^{-1}(\sqrt{3})-\tan^{-1}(1)=\pi/3-\pi/4=\pi/12$$

So the correct answer is **E**, namely, $\pi/12$.

7. Integrate:

$$\int_0^1 \frac{t^3}{\sqrt{16-4t^8}}\,dt$$

A.) $\pi/48$ B.) $\pi/24$ C.) $\pi/16$ D.) $\pi/12$ E.) $\pi/8$ F.) $\pi/4$

Answer. We hope to make the denominator have the form $\sqrt{1-u^2}$. So we first write

$$\int \frac{t^3}{\sqrt{16-4t^8}}\,dt = \int \frac{t^3}{\sqrt{16}\sqrt{1-\frac{4t^8}{16}}}\,dt = \frac{1}{4}\int \frac{t^3}{\sqrt{1-\frac{t^8}{4}}}\,dt$$

Then we notice that $\frac{t^8}{4} = \left(\frac{t^4}{2}\right)^2$, so we write $u = \frac{t^4}{2}$ and $du = 2t^3\,dt$, so $\frac{1}{2}\,du = t^3\,dt$. Our integral becomes

$$\frac{1}{8}\int \frac{1}{\sqrt{1-u^2}}\,du = \frac{1}{8}\sin^{-1}(u) + C = \frac{1}{8}\sin^{-1}\left(\frac{t^4}{2}\right) + C$$

We conclude

$$\int_0^1 \frac{t^3}{\sqrt{16-4t^8}}\,dt = \frac{1}{8}\sin^{-1}\left(\frac{t^4}{2}\right)\Bigg|_{t=0}^{1} = \frac{1}{8}\left(\sin^{-1}(1/2) - \sin^{-1}(0)\right) = \frac{1}{8}\left(\frac{\pi}{6} - 0\right) = \pi/48$$

So the correct answer is **A**, namely, $\pi/48$.

8. Integrate:

$$\int_1^e \frac{(\ln x)^2}{x}\,dx$$

A.) 1/3 B.) 1/2 C.) 1 D.) $\pi/2$ E.) e F.) e^3

Answer. We use u-substitution to make the problem easier. We write $u = \ln x$ and $du = \frac{1}{x}\,dx$. So we get

$$\int \frac{(\ln x)^2}{x}\,dx = \int u^2\,du = \frac{u^3}{3} + C = \frac{(\ln x)^3}{3} + C$$

We conclude

$$\int_1^e \frac{(\ln x)^2}{x}\,dx = \left.\frac{(\ln x)^3}{3}\right|_{x=1}^{e} = \frac{(\ln e)^3}{3} - \frac{(\ln 1)^3}{3} = \frac{1}{3} - \frac{0}{3} = \frac{1}{3}$$

So the correct answer is **A**, namely, 1/3.

9. Find the limit:

$$\lim_{x\to 0} \frac{\sin(x) - x}{x^3}$$

A.) -1 B.) $-1/6$ C.) 0 D.) $1/3$ E.) $1/2$ F.) 1

Answer. Evaluating the limit directly yields 0/0, so we use L'Hospital's rule, which gives us

$$\lim_{x\to 0} \frac{\sin(x) - x}{x^3} = \lim_{x\to 0} \frac{\cos(x) - 1}{3x^2}$$

Again, evaluating the limit directly yields 0/0, so we use L'Hospital's rule again:

$$\lim_{x\to 0} \frac{\cos(x) - 1}{3x^2} = \lim_{x\to 0} \frac{-\sin(x)}{6x}$$

Once more, evaluating the limit directly yields 0/0, so we use L'Hospital's rule one more time:

$$\lim_{x\to 0} \frac{-\sin(x)}{6x} = \lim_{x\to 0} \frac{-\cos(x)}{6} = -1/6$$

So the correct answer is **B**, namely, $-1/6$.

10. Find the limit:

$$\lim_{x\to 0^+} \big(\ln(\sin(2x)) - \ln x\big)$$

A.) 0 B.) 1 C.) $-1+\ln 2$ D.) $\ln 2$ E.) $\sin 2$ F.) $2\cos 2$

Answer. We first simplify, writing

$$\ln(\sin(2x)) - \ln x = \ln\left(\frac{\sin(2x)}{x}\right)$$

As $x \to 0^+$, we see that $\lim_{x\to 0^+} \frac{\sin(2x)}{x}$ is $0/0$, so we use L'Hospital's Rule, which gives us

$$\lim_{x\to 0^+} \frac{\sin(2x)}{x} = \lim_{x\to 0^+} \frac{2\cos(2x)}{1} = 2$$

So as $x \to 0^+$, we have $\frac{\sin(2x)}{x} \to 2$, and thus $\ln\left(\frac{\sin(2x)}{x}\right) \to \ln 2$.

So the correct answer is **D**, namely, $\ln 2$.