How to Decide Whether a Series Converges or Diverges

1. First, always see what happens to the term a_n as n gets very large. In particular: do the terms a_n in the series converge to zero as $n \to \infty$? If *not*, then the series is *divergent* (**Divergence Test**).

   Note: If a_n *does* converge to zero, there is no immediate conclusion, and we move on to step two.

2. Next, if the series happens to be an alternating series (with alternating signs, $+, -, +, -, \cdots$, in the terms of the series), and the terms a_n in the sequence *do* converge to zero as $n \to \infty$ (as you just checked in step 1), then the series is *convergent* (**Alternating Series Test**).

   Note 1: This conclusion *only* works if the series has alternating signs.

   Note 2: Strictly speaking, you must check one more thing here, which is that $|a_{n+1}| \le |a_n|$, i.e., that the terms are decreasing in absolute value. But this is almost always (but not necessarily!) the case if $\lim a_n = 0$.

3. Know your standard sequences. The **p-series**

   $$\sum^{\infty} \frac{1}{n^p}$$

   converges if $p > 1$, and diverges if $p \le 1$ (including $p = 1$). The **geometric series**

   $$\sum^{\infty} ar^{n-1} \text{ , or } \sum^{\infty} ar^n$$

   converges if $|r| < 1$ (r can be negative, but $-1 < r < 1$), and diverges if $|r| > 1$.

   Note 1: The p-series are mostly used in the *comparison test* (see below).

   Note 2: In a geometric series, you may have to manipulate the formulas a little to see what the value of r is. For example

   $$\sum_{n=1}^{\infty} \frac{(-10)^{n+1}}{2^{3n-2}} = \sum \frac{-10}{2^{-2}} \cdot \frac{(-10)^n}{2^{3n}} = \sum -40 \cdot \left(\frac{-10}{2^3}\right)^n .$$

   So we have a geometric series $\sum ar^n$ with $r = -\frac{10}{8} = -\frac{5}{4}$. In particular, $|r| > 1$, so the series diverges.

4. In step 1, you checked what happens to a_n as n gets large and goes to ∞. Even if the terms do not converge to zero, it is often useful to know what they are *approximately*, for large values of n. You can often use the **Comparison Test** to compare $\sum a_n$ to a p-series.

   For example, for large n the term $a_n = \frac{n+\sqrt{n}}{n\sqrt{n}}$ is approximately equal to $\frac{n}{n\sqrt{n}} = \frac{1}{\sqrt{n}}$. Therefore we can compare the two series

   $$\sum^{\infty} \frac{n+\sqrt{n}}{n\sqrt{n}} \text{ , and } \sum^{\infty} \frac{1}{\sqrt{n}}.$$

   Notice that the second series is a p-series with $p = \frac{1}{2}$, which is *divergent.*

To verify that we can really do the comparison, we divide the terms a_n in the first series by the terms b_n in the second series, and see what happens as $n \to \infty$,

$$c = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n+\sqrt{n}}{n\sqrt{n}} \cdot \frac{\sqrt{n}}{1} = \lim_{n\to\infty} \frac{n+\sqrt{n}}{n} = \lim_{n\to\infty} 1 + \frac{1}{\sqrt{n}} = 1.$$

If this limit $c = \lim a_n/b_n$ is *finite* and *positive* (!), then we can compare the two series in the following sense: if $\sum b_n$ converges, then so does $\sum a_n$, and if $\sum b_n$ diverges, then so does $\sum a_n$.

Note 1: If we find the limit $c = 0$, or $c = \infty$, then the two series can not be compared!

Note 2: The comparison test works only if both series have *positive* terms.

5. If the formula for a_n is a function of n that can be easily integrated, use the **Integral Test**. This test says that, if $a_n = f(n)$, and f is a *positive* but *decreasing* function, then if the integral $\int_1^\infty f(x)dx$ converges (to a finite value), then so does the series $\sum a_n$, and if the integral diverges, so does the series.

   For example, the series $\sum n^2 e^{-n}$ converges, because the integral $\int_1^\infty x^2 e^{-x}$ converges (use partial integration).

6. If the formula for the term a_n contains *factorials* $n!$, or powers with n (or expressions like $2n-3$, etc.) as an *exponent*, you can often use the **Ratio Test**. You must evaluate the limit

   $$L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}.$$

   If this limit L exists, then we have the following rule: (i) if $L < 1$ the series is absolutely convergent, (ii) if $L > 1$ the series is divergent, and (iii) if $L = 1$ there is no conclusion.

   Note: To find L we take the *absolute values* of a_{n+1} and a_n, i.e., this test works regardless of the signs of the terms in the series.

7. If the terms a_n can be expressed as a single power with exponent n (or a linear expression like $2n+3$ etc.), use the **Root Test**. It works the same as the Ratio Test, except that you find L by evaluating the limit

   $$\lim_{n\to\infty} \sqrt[n]{|a_n|}.$$