Math 241, Fall 2004
Homework Assignment #1

1. Compute the Fourier series of the function f of period 2 and defined by $f(x) = x^2$ for $-1 \leq x \leq 1$.

2. Compute the Fourier series of the function f of period 2π given by

$$f(x) = \begin{cases} 1 & \text{for } -\pi \leq x \leq 0 \\ 2 & \text{for } 0 < x < \pi \end{cases}$$

3. Compute the Fourier series of the 2π-periodic function given by $f(x) = \sin^2 x$ on $-\pi \leq x \leq \pi$.
   Hint: You don't need to use any calculus.

4. Compute the Fourier series of the 2π periodic function given by $f(x) = |\sin x|$ on $-\pi \leq x \leq \pi$.

5. Suppose

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$g(x) \sim \frac{1}{2}c_0 + \sum_{n=1}^{\infty}(c_n \cos nx + d_n \sin nx)$$

   (a) What is the Fourier series of $h(x) = f(x) + g(x)$?
   (b) What is the Fourier series of $h(x) = cf(x)$, where c is a constant?
   (c) Assume that f and f' are continuous and 2π periodic what is the Fourier series for $f'(x)$?
   Hint: Integrate by parts.

6. Suppose $f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} \sin nx$ what is

   (a)
$$\int_{-\pi}^{\pi} f(x) \sin 10x \, dx$$

   (b)
$$\int_{-\pi}^{\pi} f(x) \cos 9x \, dx$$

7. Compute

$$\int_{-\pi}^{\pi} (\sin 7x + 2\sin 9x - 5\cos 2x)(\sin 3x + \sin 100x - \cos 2x + \cos 4x + 5) \, dx$$

   Hint: Think orthogonality and this does not require much computation.

8. If f has period p show

$$\int_0^p f(x)\,dx = \int_T^{T+p} f(x)\,dx$$

for any constant T. Now compute

$$\int_{7\pi}^{9\pi} \sin 100x \cos 957x\,dx$$

9. Suppose f is 2π periodic.

   (a) Suppose f' are continuous and 2π periodic. Suppose a_n and b_n are the Fourier coefficients for f. Show that

   $$\lim_{n\to\infty} a_n = 0$$

   and

   $$\lim_{n\to\infty} b_n = 0$$

   Hint: Look back at the proof that Fourier series converge we did in class. Then try to show $|a_n| \leq C\frac{1}{n}$ for some constant C.

   (b) Suppose f'' are continuous and 2π periodic. Suppose a_n and b_n are the Fourier coefficients for f. Show that

   $$\lim_{n\to\infty} na_n = 0$$

   and

   $$\lim_{n\to\infty} nb_n = 0$$

   Remark: This implies that the smoothness (that is how many derivatives a function has) is related to how fast its Fourier coefficients go to 0.

   (c) Does $\sum \frac{1}{n} \sin nx$ converge to a function with two continuous derivatives?