Math 241, Fall 2004
Homework Assignment #11

1. Suppose C is a simple closed curve and a and b are complex numbers not on C. What are all the possible values of

$$\int_C \frac{1}{(z-a)(z-b)}\, dz?$$

   HINT: You should determine all the possible locations of the points a and b relative to C.

2. Compute $\int_C \frac{e^z}{z}\, dz$ where C is $|z| = 1$. Use your answer to show that $\int_0^\pi e^{\cos t} \cos(\sin t)\, dt = \pi$. Notice you have used a complex contour integral to compute an integral of a function of a real variable.
   HINT: Parameterize C by $z(t) = -e^{it}$, where $-\pi \leq t \leq \pi$.

3. Suppose f and g are analytic inside a simple closed curve C. Suppose $f(z) = g(z)$ for all z on C. Show $f(z) = g(z)$ for all z inside C.

4. Suppose f is entire and that its image misses all numbers in some disk. (This means that there is some z_0 and some ρ such that for all z, $f(z)$ is never in the disk of radius ρ about z_0.) Show f must be constant.
   HINT: Consider $g(z) = \frac{1}{f(z)-z_0}$ and show you can apply Liouville's theorem.

5. Find the radius of convergence for the Taylor series of the following functions about the given point z_0. (You do not need to find the Taylor series to answer this question.)
   (a) $f(z) = \frac{1}{z-5i}$ about $z_0 = 3$.
   (b) $f(z) = \frac{1}{\sin z}$ about $z_0 = \frac{\pi}{2}$.
   (c) $f(z) = \frac{1}{e^z}$ about $z_0 = 7$.

6. Find the Taylor series for the following functions about the given point. Determine the radius of convergence. (For the following problems you do not need to use the formula for the coefficients of the Taylor series. If you use the formula it will require much more work.)
   (a) $f(z) = e^{z^2}$ about $z_0 = 0$.
   (b) $f(z) = e^z$ about $z_0 = 2$.
   (c) $f(z) = z^2 e^z$ about $z_0 = 0$.
   (d) $f(z) = 2ze^z + z^2 e^z$ about $z_0 = 0$.
   HINT: Differentiate the answer from the previous problem
   (e) $f(z) = \frac{\sin z}{z}$ about $z_0 = 0$.
   (f) $f(z) = \frac{1}{1-z}$ about $z_0 = 10$.

7. The Bessel function of order n is defined by

$$J_n(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+n)!} \left(\frac{1}{2}\right)^{2k+n} z^{2k+n}$$

   (a) Show $J_n(z)$ is entire for each n.
   (b) Show that $\frac{d}{dz}\left[z^n J_n(z)\right] = z^n J_{n-1}(z)$

8. Find the Laurent expansions of the following functions in the given regions.
   (a) $f(z) = \frac{\sin z}{z^2}$ for $|z| > 0$.
   (b) $f(z) = \frac{3+z}{2-z}$ for $2 < |z|$.
   (c) $f(z) = \frac{4z-5}{(z-2)(z-1)}$ for $|z-2| > 1$.

9. Find all Laurent expansions of $f(z) = \frac{1}{z^2+1}$ around $z_0 = 1$.

10. Find the Laurent expansion for $f(z) = \frac{z^2}{(1+z)^4}$ for $|z| > 1$.
HINT: Think about $\frac{1}{1+z}$ and differentiation.