Math 241, Fall 2004
Homework Assignment #12

1. Determine the zeros of the following functions and the order of the zero.

   (a) $f(z) = (z - 3 + i)^3$
   (b) $f(z) = z + \frac{16}{z}$
   (c) $f(z) = \sin^2 z$
   (d) $f(z) = z - \sin z$

2. Find all the singularities of the following functions. For the isolated singularities determine if they are removable (in which case show how to remove the singularity), poles (in which case determine their order) or essential.

   (a) $f(z) = \frac{e^{2z}-1}{z}$
   (b) $f(z) = \frac{e^z-1}{z^3}$
   (c) $f(z) = \frac{z}{\sin z}$
   (d) $f(z) = \tan(z^2)$
   (e) $f(z) = \frac{1}{e^{z^2}}$
   (f) $f(z) = e^{\frac{1}{z^3}}$
   (g) $f(z) = \tan(\frac{1}{z})$
   HINT: $z = 0$ is not an isolated singularity.

3. Compute the residues of the following functions at each of their singularities.

   (a) $f(z) = \frac{7}{(z-2)(z+i)}$
   (b) $f(z) = \frac{2}{z^2(z-3)^3}$
   (c) $f(z) = e^{\frac{2}{z}}$
   (d) $f(z) = \frac{1}{\sin z}$
   (e) $f(z) = \frac{e^z}{e^{2z}-1}$

4. Use the residue theorem to compute the following integrals. (All curves oriented counter clockwise.

   (a) $\int_C \frac{3}{z(z-1)}\,dz$ where C is $|z| = 2$
   (b) $\int_C \frac{3}{z(z-1)}\,dz$ where C is $|z-1| = \frac{1}{2}$
   (c) $\int_C \frac{1}{z^4+1}\,dz$ where C is $|z-1| = \sqrt{2}$
   (d) $\int_C \frac{ze^z}{z^2-1}\,dz$ where C is $|z| = 2$
   (e) $\int_C \frac{\tan z}{z}\,dz$ where C is $|z-1| = 2$

5. Show that if $f(z)$ is analytic near z_0 and has a zero of order m at z_0 then there is a function $g(z)$ that is analytic near z_0 such that $g(z_0) \neq 0$ and $f(z) = (z - z_0)^m g(z)$ for z near z_0.

6. Suppose $f(z)$ is analytic near z_0 and has an isolated zero of order m at z_0. Show that the function $\frac{f'(z)}{f(z)}$ has a simple pole at z_0 and the residue there is m.
HINT: use the previous problem.

7. Suppose $f(z)$ is analytic inside and on a curve C and has only isolated zeros inside C (and none on C). Let $N(f, C)$ be the number of zeros of f inside C counted with multiplicity. Show
$$N(f, C) = \frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)}\, dz.$$
You may assume there are only a finite number of zeros (this follows from the hypothesis on f but you don't need to show it.)

8. Suppose g and h are analytic near z_0, $g(z_0) \neq 0$ and h has a zero of order 2 at z_0. Show
$$\mathrm{Res}(\frac{g(z)}{h(z)}, z_0) = 2\frac{g'(z_0)}{h''(z_0)} - \frac{2}{3}\frac{g(z_0)h'''(z_0)}{(h''(z_0))^2}.$$
HINT: We did something like this in class.

9. Compute the following integrals.
   (a) $\int_0^{2\pi} \frac{1}{5+3\cos\theta}\, d\theta$.
   (b) $\int_0^{\pi} \frac{1}{10+8\cos\theta}\, d\theta$.
   HINT: Use the evenness and periodicity of the integrand to change the limits of integration to 0 to 2π.

10. If $z = e^{i\theta}$ then show
$$\cos n\theta = \frac{1}{2}(z^n + \frac{1}{z^n})$$
and
$$\sin n\theta = \frac{1}{2i}(z^n - \frac{1}{z^n}).$$

11. Use problem 10 to compute
$$\int_0^{2\pi} \frac{\cos 2\theta}{2 + \cos\theta}\, d\theta.$$

12. Compute the following integrals.
   (a) $\int_{-\infty}^{\infty} \frac{1}{x^4+1}\, dx$.
   (b) $\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^4+1)}\, dx$.