Math 241, Fall 2004
Homework Assignment #5

1. Solve
$$u_t = 5u_{xx}$$
$$u(0,t) = 0, \quad u(8,t) = 0$$
$$u(x,0) = \sin\frac{\pi}{4}x + \sin 8\pi x, \quad 0 \le x \le 8.$$

2. Solve
$$u_t = 5u_{xx}$$
$$u(0,t) = 7, \quad u(8,t) = 47$$
$$u(x,0) = \sin\frac{\pi}{4}x + 5x + 7.$$

3. Solve
$$u_t = 2u_{xx}$$
$$u_x(0,t) = 0, \quad u_x(5,t) = 0$$
$$u(x,0) = x, \quad 0 \le x \le 5.$$

4. Let $u(x,t)$ be a solution to the heat equation
$$u_t = ku_{xx}$$
Show the following hold.
   (a) For constants a, x_0, t_0 the function $v(x,t) = u(ax - x_0, a^2t - t_0)$ also solves the above heat equation.
   (b) The function $v(x,t) = t^{-\frac{1}{2}}e^{\frac{-x^2}{4kt}}u(\frac{x}{t}, -\frac{1}{t})$ solves the above heat equation.
   (c) For any constant k' the function $v(x,t) = u(x, \frac{k'}{k}t)$ solves the equation $v_t = k'v_{xx}$.

5. Consider the problem
$$u_t = c^2u_{xx}, \quad -\pi \le x \le \pi,$$
$$u(\pi,t) = u(-\pi,t), \quad u_x(\pi,t) = u_x(-\pi,t)$$
$$u(x,0) = f(x).$$
Find a formula for the solution in the following steps.
   (a) Use separation of variable. That is assume the solution can be written $u(x,t) = X(x)T(t)$ and derive ODE's for $X(x)$ and $T(t)$.
   (b) Solve the ODE's from (a) so that your solutions also satisfy the second equation.
   (c) Write down the most general solution to the original equation coming from separation of variables. Then determine how to satisfy the last equation.

6. Solve

$$u_{tt} = a^2 u_{xx}, \quad 0 \le x \le L$$
$$u(0,t) = 0, \quad u(L,t) = 0$$
$$u(x,0) = f(x), \quad u_t(x,0) = g(x)$$

in the following cases.

(a) $f(x) = 2\sin\frac{\pi}{L}x - \sin\frac{4\pi}{L}x$ and $g(x) = \frac{1}{2}\sin\frac{2\pi}{L}x$

(b) $f(x) = \sin^3\frac{\pi}{L}x$ and $g(x) = 0$.
HINT: use a trigonometric identity.

(c) $f(x) = x(L-x)$ and $g(x) = 0$

7. Consider the problem

$$u_{xx} = u_{tt} + 2ku_t, \quad 0 < k < 1$$
$$u(0,t) = 0, \quad u(\pi,t) = 0$$
$$u(x,0) = f(x), \quad u_t(x,0) = 0.$$

Find a formula for the solution in the following steps.

(a) Use separation of variable. That is assume the solution can be written $u(x,t) = X(x)T(t)$ and derive ODE's for $X(x)$ and $T(t)$.

(b) Solve the ODE's from (a) so that your solutions also satisfy the second equation.

(c) Write down the most general solution to the original equation coming from separation of variables. Then determine how to satisfy the last equation.