Math 241, Fall 2004
Homework Assignment #6

1. Find $u(x, y, t)$ solving $u_t = c^2 \nabla^2 u$ in the region $0 < x < 2, 0 < y < 5$ and $t > 0$, such that $u(x, y, t) = 0$ on the boundary of the region and

$$u(x, y, 0) = -10(\sin 3\pi x)(\sin 2\pi y).$$

2. Find $u(x, y)$ so that it solves

$$u_{xx} + u_{yy} = 0, \quad 0 < x < a, 0 < y < b$$

$$u(0, y) = 0, \quad u(a, y) = 0$$

$$u(x, 0) = 0$$

$$u(x, b) = f(x)$$

in the following steps.

   (a) Use separation of variable. That is assume the solution can be written $u(x, y) = F(x)G(y)$ and derive ODE's for $F(x)$ and $G(y)$.

   (b) Solve the ODE for $F(x)$ so that $F(x)$ also satisfies the second equation above.

   (c) Solve the ODE for $G(y)$ so that $G(y)$ also satisfies the third equation above. (Ignore the fourth equation for now.)
   Your solutions for $G(y)$ should be

$$G_n(y) = A_n(e^{\frac{n\pi}{a}y} - e^{-\frac{n\pi}{a}y}).$$

   There is a shorthand notation for this, the hyperbolic sine is defined to be

$$\sinh x = \frac{e^x - e^{-x}}{2}.$$

   So we can write $G_n(y) = \sinh \frac{n\pi}{a} y$.

   (d) Write down the most general solution to the original equation coming from separation of variables. Then determine how to satisfy the last equation.

3. Solve $u_{xx} + u_{yy} = 0$ on the rectangle $0 < x < 3, 0 < y < 2$ subject to the boundary conditions

   (a) $u(0, y) = u(3, y) = 0, \quad u(x, 0) = 0, \quad u(x, 2) = 5 \sin \pi x - 3 \sin 10\pi x$

   (b) $u(0, y) = u(3, y) = 0, \quad u(x, 0) = 0, \quad u(x, 2) = 10$

4. Solve $\nabla^2 u = 0$ in the region $r < 7$ (we are using polar coordinates here) and

$$u(7, \theta) = -\theta^2 + 2\pi\theta.$$

5. Show that in spherical coordinates

$$\nabla^2 u(r,\phi,\theta) = u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + \frac{\cot\theta}{r^2}u_\theta + \frac{1}{r^2\sin^2\theta}u_{\phi\phi}$$

(You can show this exactly like we did the formula for $\nabla^2 u$ in polar coordinates in class.)

6. Given functions $f(x)$ and $g(x)$ show that

$$u(x,t) = \frac{1}{2}(f(x+at) + f(x-at)) + \frac{1}{2a}\int_{x-at}^{x+at} g(s)\,ds$$

solves

$$\begin{aligned} u_{tt} &= a^2 u_{xx} \\ u(x,0) &= f(x) \\ u_t(x,0) &= g(x), \end{aligned}$$

for all $-\infty < x < \infty$ and $-\infty < t < \infty$. It might be helpful to recall

$$\frac{\partial}{\partial x}\int_{a(x)}^{b(x)} h(s)\,ds = h(b(x))b'(x) - h(a(x))a'(x).$$

This form of the solution to the wave equation is called D'Alembert's solution.

7. Using problem 6 solve the wave equation (with notation as in problem 6) in the following cases.

   (a) $f(x) = e^{-x^2}, \quad g(x) = 0$

   (b) $f(x) = \sin x, \quad g(x) = 1$

   (c) $f(x) = 0, \quad g(x) = xe^{-x^2}$

   (d) You do not have to turn this in, but think about the solution to (a). Try to draw the graph for various times.

8. Prove that if u_1 and u_2 solve

$$u_{tt} = a^2 u_{xx}, \quad 0 \le x \le L, -\infty < t < \infty$$

$$u(0,t) = A(t), \quad u(L,t) = B(t)$$

$$u(x,0) = f(x), \quad u_t(x,0) = g(x)$$

then $u_1(x,t) = u_2(x,t)$ for all x and t. Do this in the following steps. (These are almost exactly like what we did in class for the heat equation.)

   (a) Set $v(x,t) = u_1(x,t) - u_2(x,t)$. Show that $v(x,t)$ satisfies

$$v_{tt} = a^2 v_{xx}$$

$$v(0,t) = 0, \quad v(L,t) = 0$$

$$v(x,0) = 0, \quad v_t(x,0) = 0.$$

(b) Consider the "energy of v" given by

$$H(t) = \int_0^L [a^2 v_x^2(x,t) + v_t^2(x,t)]dx.$$

Show $H(0) = 0$.

(c) Show $H'(t) = 0$. Do this by providing justification for the following equalities.

$$\begin{aligned} H'(t) &= \int_0^L [a^2 2 v_x v_{xt} + 2 v_t v_{tt}]dx \\ &= 2a^2 \int_0^L [v_x v_{xt} + v_t v_{xx}]dx \\ &= 2a^2 \int_0^L \frac{\partial}{\partial x}[v_x v_t]dx \\ &= 2a^2 [v_x(x,t) v_t(x,t)]|_0^L \\ &= 0 \end{aligned}$$

(d) Conclude that $H(t) = 0$ and thus $v_t(x,t) = 0$ for all t and x.

(e) Explain how this implies $v(x,t) = 0$ and hence $u_1(x,t) = u_2(x,t)$.