Math 241 Practice Final Answers

1) True False

1. True. Period $2\pi i$
2. False.
3. True.
4. False.
5. False.
6. False. It does not satisfy the Cauchy-Riemann Equations.
7. False. The radius of convergence is 5.
8. True.

2) Short Answer

1. 12.
2. $2\pi i$.
3. $e^{i\frac{\pi}{4}}, e^{i\frac{3\pi}{4}}, e^{i\frac{5\pi}{4}}, e^{i\frac{7\pi}{4}}$.
4. Essential.
5. 0. The function is analytic.
6. 7. All the Fourier coefficients are zero except for 7, the coefficient on $\cos 2x$.
7. $\sqrt{2}$.
8. 3.
9. -4.

3)

1. The integrand has simple poles at $z = i, -i$. Both poles are inside the contour C.

$$\text{Res}(\frac{z^2-4}{z^2+1}, i) = \frac{i^2-4}{2i} = \frac{-5}{2i} = \frac{5}{2}i.$$

$$\text{Res}(\frac{z^2-4}{z^2+1}, -i) = \frac{(-i)^2-4}{-2i} = \frac{-5}{-2i} = -\frac{5}{2}i.$$

So

Answer:

$$\int_C \frac{z^2-4}{z^2+1}\, dz = 2\pi i(\frac{5}{2}i - \frac{5}{2}i) = 0$$

2. The integrand has a pole of order 2 at 2. The pole is inside the contour C.

$$\text{Res}(\frac{e^z}{(z-2)^2}, 2) = \lim_{z\to 2} \frac{d}{dz}[(z-2)^2 \frac{e^z}{(z-2)^2}] = \lim_{z\to 2} e^z = e^2.$$

So

Answer:

$$\int_C \frac{e^z}{(z-2)^2}\, dz = 2\pi i(e^2) = 2e^2\pi i.$$

3. The integrand, which we think of as $\frac{\sin z}{z\cos z}$, has simple poles at 0 and all odd multiples of $\frac{\pi}{2}$. The poles inside C are $\frac{\pi}{2}$, $\frac{3\pi}{2}$, and $\frac{5\pi}{2}$. Note that $z = 0$ is a removable singularity and hence its residue is 0.

$$\text{Res}(\frac{\sin z}{z\cos z}, \frac{\pi}{2}) = \frac{1}{0-(\frac{\pi}{2})} = -\frac{2}{\pi}$$

$$\text{Res}(\frac{\sin z}{z\cos z}, \frac{3\pi}{2}) = \frac{-1}{0-(\frac{3\pi}{2})(-1)} = -\frac{2}{3\pi}$$

Similarly $\text{Res}(\frac{\sin z}{z\cos z}, \frac{3\pi}{2}) = -\frac{2}{5\pi}$. So

Answer:

$$\int_C \frac{\tan z}{z}\, dz = -2\pi i(\frac{2}{\pi} + \frac{2}{3\pi} + \frac{2}{5\pi}) = -2\pi i\frac{30+10+6}{15\pi} = -\frac{92}{15}i.$$

4)

1. $f(z)$ has three different Laurent expansions about $z=1$. One good in $|z-1|<1$ one good in $1<|z-1|<4$ and one good in $|z-1|>4$.

2. Using partial fractions you find

$$f(z) = -\frac{\frac{1}{5}}{z+3} + \frac{\frac{1}{5}}{z-2}$$

Recall $\frac{1}{1-w} = \sum_{n=0}^{\infty} w^n$ good for $|w|<1$. So

$$\frac{1}{4+(z-1)} = \frac{1}{4(1+\frac{z-1}{4})} = \frac{1}{4}\sum_{n=0}^{\infty}(-\frac{1}{4})^n(z-1)^n = -\sum_{n=0}^{\infty}(-\frac{1}{4})^{n+1}(z-1)^n$$

Similarly $\frac{1}{1-w} = \sum_{n=0}^{\infty} -(\frac{1}{w})^{n+1}$ for $|w|>1$. So

$$\frac{1}{-1+(z-1)} = \frac{1}{-(1-(z-1))} = -\sum_{n=0}^{\infty} -(z-1)^{-(n+1)} = \sum_{n=0}^{\infty}(z-1)^{-(n+1)}$$

Thus

Answer:

$$f(z) = \sum_{n=0}^{\infty}\frac{1}{5}(-\frac{1}{4})^{n+1}(z-1)^n + \sum_{n=0}^{\infty}\frac{1}{5}(z-1)^{-(n+1)}$$

5)

$$\begin{aligned} b_n &= \frac{2}{3}\int_0^3 (3x-4)\sin\frac{n\pi}{3}x\,dx \\ &= \frac{2}{3}\left(-\frac{3}{n\pi}(3x-4)\cos\frac{n\pi}{3}x + \frac{9}{n^2\pi^2}3\sin\frac{n\pi}{3}x\right)\Big|_0^3 \\ &= \frac{2}{3}(-\frac{15}{n\pi}(-1)^n + 0 + \frac{-12}{n\pi} - 0) = -\frac{2}{3n\pi}(12+(-1)^n 15) \end{aligned}$$

So the Fourier sine expansion is

Answer:

$$\sum_{n=1}^{\infty} -\frac{2}{3n\pi}(12+(-1)^n 15)\sin\frac{n\pi}{3}x$$

6) 1. Make the substitution $z=e^{ix}$ so $dx = \frac{1}{iz}dz$ and $\cos x = \frac{1}{2}(z+\frac{1}{x})$. Thus if C is the unit circle our integral becomes

$$\int_C \frac{1}{2-\frac{1}{2}(z+\frac{1}{z})}\frac{1}{iz}dz = 2i\int_C \frac{1}{z^2-4z+1}\,dz.$$

The integrand has simple poles at $z = 2 \pm \sqrt{3}$, but only $2 - \sqrt{3}$ is inside the contour C. Thus

$$\int_C \frac{1}{z^2 - 4z + 1}\, dz = 2\pi i(\text{Res}(\frac{1}{z^2 - 4z + 1}, 2 - \sqrt{3}) = 2\pi i \frac{1}{2(2 - \sqrt{3}) - 4} = -\frac{\pi i}{\sqrt{3}}.$$

Thus

Answer:

$$2i(-\frac{\pi i}{\sqrt{3}}) = \frac{2\pi}{\sqrt{3}}.$$

2. According to a theorem from class we know the value of this integral is the value of $\lim_{R\to\infty} \int_{C_R} \frac{z^2}{z^4+1}\, dz$ where C_R is the upper half circle of radius R about the origin plus the part of the x-axis necessary to make a closed curve. By the Residue Theorem this integral is simple $2\pi i$ times the sum of the residues of the singularities of the integrand in the upper half plane. The singularities of the integrand are at the 4^{th} roots of unity, that is $\pm\frac{\sqrt{2}}{2} \pm i\frac{\sqrt{2}}{2}$. The ones in the upper half plane are $\pm\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$. Since these singularities are simple poles we can compute that the residues are

$$\text{Res}(\frac{z^2}{z^4+1}, \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}) = \frac{z^2}{4z^3}\Big|_{z=\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}} = \frac{(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2})^2}{4(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2})^3} = \frac{1}{4(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2})} =$$
$$\frac{\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}}{4} = \frac{\sqrt{2}}{8} - i\frac{\sqrt{2}}{8}$$

Similarly

$$\text{Res}(\frac{z^2}{z^4+1}, -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}) = -\frac{\sqrt{2}}{8} - i\frac{\sqrt{2}}{8}$$

Thus

Answer:

$$\int_{-\infty}^{\infty} \frac{x^2}{x^4+1}\, dx = 2\pi i[(\frac{\sqrt{2}}{8} - i\frac{\sqrt{2}}{8}) + (-\frac{\sqrt{2}}{8} - i\frac{\sqrt{2}}{8})] = \frac{\pi\sqrt{2}}{2}$$

7) Take the Laplace transform of both sides to get

$$\mathcal{L}\{y'' - 5y' + 4y = 0\} = 0$$

so

$$s^2\hat{y} - sy(0) - y'(0) - 5s\hat{y} + 5y(0) + 4\hat{y} = 0$$

$$(s^2 - 5s + 4)\hat{y} = 2$$
$$\hat{y} = \frac{2}{s^2 - 5s + 4} = \frac{2}{3}\frac{1}{s-4} - \frac{2}{3}\frac{1}{s-1}.$$

Thus
$$y(t) = \mathcal{L}^{-1}\{\hat{y}\} = \frac{2}{3}\mathcal{L}^{-1}\{\frac{1}{s-4}\} - \frac{2}{3}\mathcal{L}^{-1}\{\frac{1}{s-1}\}.$$

So

Answer:
$$y(t) = \frac{2}{3}e^{4t} - \frac{2}{3}e^{t}.$$

8) 1. Let $f(z) = u(x, y) + iv(x, y)$ be an analytic function. The real part of f is $u(x, y)$.

$$u_{xx} + u_{yy} = (u_x)_x + (u_y)_y = (v_y)_x + (-v_x)_y = v_{xy} - v_{xy} = 0,$$

where the second equality follows from the Cauchy Riemann equations.
2. The function $u(x, y) = xy - 1$ is clearly harmonic. If $v(x, y)$ is its harmonic conjugate then $v_y = u_x = y$ and $v_x = -u_y = -x$. Integrating the first equation with respect to y gives

$$v(x, y) = \frac{1}{2}y^2 + g(x).$$

Thus $v_x = g'(x)$ but this must be $-x$ so $g(x) = -\frac{1}{2}x^2 + c$. So

Answer:
$$v(x, y) = \frac{1}{2}(y^2 - x^2 + c)$$

9) The equation becomes

$$X''T - 2XT'' + 7XT' + XY = 0.$$

Dividing by XT and rearranging we get

$$\frac{X''}{X} = \frac{1}{T}(2T'' - 7T') - 1.$$

Since each side is a function of a different variable they must both be constant. Let the constant be k. So we get equations for X and T:

$$X'' - kX = 0$$

$$2T'' - 7T' - (k+1)T = 0$$

The boundary conditions for u imply $X(0) = 0$ and $X(7) = 0$. As we have seen many times this leads to solutions

$$X_n(x) = \sin(\frac{n\pi}{7}x).$$

Plugging into the equation for T give

$$T_n(t) = a_n e^{\lambda_n t} + b_n e^{\lambda'_n t}$$

where $\lambda_n = \frac{7+\sqrt{49+8(1-(\frac{n\pi}{7})^2)}}{4}$ and $\lambda'_n = \frac{7-\sqrt{49+8(1-(\frac{n\pi}{7})^2)}}{4}$ when $n < 6$ and when $n \geq 6$ we get

$$T_n(t) = a_n e^{\frac{7}{4}t} \sin \lambda''_n t + b_n e^{\frac{7}{4}t} \cos \lambda''_n t$$

where $\lambda''_n = \frac{\sqrt{-49+8((\frac{n\pi}{7})^2-1)}}{4}$. Thus the solution to the PDE that can be written as the product to $X(x)$ and $T(t)$ are

Answer:

$$u(x,t) = \sin(\frac{n\pi}{7}x)(a_n e^{\lambda_n t} + b_n e^{\lambda'_n t}) \quad \text{for } n < 6$$

$$u(x,t) = \sin(\frac{n\pi}{7}x)(a_n e^{\frac{7}{4}t} \sin \lambda''_n t + b_n e^{\frac{7}{4}t} \cos \lambda''_n t) \quad \text{otherwise}$$

10) 1. Since $f(z)$ has an isolated 0 at z_0 we can look at its Laurent expansion about z_0

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n.$$

A piece of this expansion looks like

$$\ldots + a_{-2}\frac{1}{(z-z_0)^2} + a_{-1}\frac{1}{z-z_0} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \ldots$$

If we differentiate the series for f we get the series for f', a part of which is

$$\ldots - 2a_{-2}\frac{1}{(z-z_0)^3} - a_{-1}\frac{1}{(z-z_0)^2} + a_1 + 2a_2(z-z_0) + \ldots.$$

So we see the coefficient on the $\frac{1}{z-z_0}$ term in the Laurent expansion of $f'(z)$ is 0.

2. Again we can write the Laurent expansion of $f(z)$

$$f(z) = a_{-m}\frac{1}{(z-z_0)^m} + a_{-(m-1)}\frac{1}{(z-z_0)^(m-1)} + \ldots a_{-1}\frac{1}{z-z_0} + a_0 + a_1(z-z_0) + \ldots.$$

Now factor a $(z-z_0)^{-m}$ out of the right hand side to get

$$f(z) = (z-z_0)^{-m}(a_{-m} + a_{-(m-1)}(z-z_0) + \ldots a_{-1}(z-z_0)^{m-1} + a_0(z-z_0)^m + a_1(z-z_0)^{m+1} + \ldots).$$

The stuff in the parenthesis defines an analytic function $g(z)$ (since there is no principal part to the Laurent expansion). So we have written $f(z) = (z-z_0)^{-m}g(z)$. Also note $g(0) = a_{-m} \neq 0$ since z_0 was a pole of order m of $f(z)$.

Now $f'(z) = -m(z-z_0)^{-m-1}g(z) + (z-z_0)^{-m}g'(z)$ so

$$\frac{f'(z)}{f(z)} = \frac{-m(z-z_0)^{-m-1}g(z) + (z-z_0)^{-m}g'(z)}{(z-z_0)^{-m}g(z)} = \frac{-mg(z) + (z-z_0)g'(z)}{(z-z_0)g(z)}.$$

So we see $\frac{f'(z)}{f(z)}$ has a simple pole at z_0 (note the numerator is not 0 at z_0 and the denominator has a zero of order 1). Thus

$$\text{Res}(\frac{f'(z)}{f(z)}, z_0) = \frac{-mg(z_0) + (0)g'(z_0)}{g(z_0) + 0g'(z_0))} = -m$$

11) We are looking for a power series solution $y(x) = \sum_{n=0}^{\infty} a_n x^n$. So $y'(x) = \sum_{n=1}^{\infty} na_n x^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$ and $y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$. Thus the equation is

$$y'' + 2y' + y = \sum_{n=0}^{\infty}[(n+2)(n+1)a_{n+2} + 2(n+1)a_{n+1} + a_n]x^n = 0.$$

So

Answer:

$$a_{n+2} = \frac{-a_n - 2(n+1)a_{n+1}}{(n+2)(n+1)}.$$