Math 241 Practice Midterm Answers

1) True False

1. False.
2. True.
3. False. The function $f(x) = 0$ is odd and even.
4. False. The function is even.
5. False. In fact if $P = 2L$ the function will also have period P.
6. True. If you change the value of a function at one point you will have a different function but both these functions will have the same Fourier series.

2) Short Answer

1. 0. Since $0 = a_0 = \int_{-\pi}^{\pi} f(x)\,dx$.
2. 0. The function is odd so all a_n will be zero.
3. $a_n + a_n'$ and $b_n + b_n'$ will be the Fourier coefficients of $f + g$.
4. 1. The nearest singular point to 0 is ± 1 and it has distance 1 form 0.

3) We first solve

$$y'' - y' - 6y = \frac{1}{n(n+1)} \sin nx$$

We look for a solution of the form

$$y = A \cos\, nx + B \sin\, nx.$$

Differentiating we get

$$y' = -An \sin\, nx + Bn \cos\, nx$$

$$y'' = -An^2 \cos\, nx + Bn^2 \sin\, nx.$$

Plugging into the equation above and collecting terms we get

$$(-An^2 - Bn - 6A)\cos nx + (-Bn^2 + An - 6B)\sin nx = \frac{1}{n(n+1)}\sin nx.$$

So we must have

$$-An^2 - Bn - 6A = 0$$
$$-Bn^2 + An - 6B = \frac{1}{n(n+1)}$$

Thus $B = -\left(\frac{n^2+6}{n}\right) A$ and

$$A\left(\frac{(n^2+6)^2+n^2}{n}\right) = \frac{1}{n(n+1)}$$

Finally, $A = \frac{1}{(n+1)((n^2+6)^2+n^2)}$ and $B = -\frac{n^2+6}{n(n+1)((n^2+6)^2+n^2)}$, thus the solution to the equation above is

$$y_1(x) = \frac{1}{(n+1)((n^2+6)^2+n^2)}\left(\cos nx - \left(\frac{n^2+6}{n}\right)\sin nx\right)$$

So the solution to the original problem is

Answer:

$$y(x) = \sum_{n=1}^{\infty} \frac{1}{(n+1)((n^2+6)^2+n^2)}\left(\cos nx - \left(\frac{n^2+6}{n}\right)\sin nx\right)$$

4) The function is odd so $a_n = 0$ for all n. to find the b_n's we compute

$$\begin{aligned} b_n &= \frac{1}{2}\int_{-2}^{2} x \sin \frac{n\pi}{2}x\,dx \\ &= \frac{1}{2}\left(-\frac{2}{n\pi}x\cos\frac{n\pi}{2}x|_{-2}^{2} + \int_{-2}^{2}\frac{2}{n\pi}\cos\frac{n\pi}{2}x\,dx\right) \\ &= \frac{1}{2}\left(-\frac{2}{n\pi}4\cos n\pi + \left(\frac{2}{n\pi}\right)^2 \sin\frac{n\pi}{2}x\,|_{-2}^{2}\right) \\ &= \frac{1}{2}\left(-\frac{2}{n\pi}4(-1)^n + \left(\frac{2}{n\pi}\right)^2 2\sin n\pi\right) \\ &= (-1)^{n+1}\frac{4}{n\pi} \end{aligned}$$

So we have

Answer:
$$a_n = 0, \quad b_n = (-1)^{n+1}\frac{4}{n\pi}$$

5) Compute

$$\begin{aligned} 0 &= \int_0^1 f^2 g'\,dx \\ &= f^2 g|_0^1 - \int_0^1 2ff'g\,dx \\ &= f^2(1)g(1) - f^2(0)g(0) - \int_0^1 2ff'g\,dx \\ &= 0 - \int_0^1 2ff'g\,dx \end{aligned}$$

Thus we see that $\int_0^1 ff'g\,dx = 0$ so the functions ff' and g are orthogonal.

Answer: Yes, they are orthogonal.

6) Compute

$$\begin{aligned} \int_1^2 xy_8(x)\ln x\,dx &= \int_1^2 xy_8(x)\sum_{n=1}^{\infty}\frac{1}{n^2}y_n(x)\,dx \\ &= \sum_{n=1}^{\infty}\int_1^2 xy_8(x)\frac{1}{n^2}y_n(x)\,dx \\ &= \int_1^2 xy_8(x)\frac{1}{8^2}y_8(x)\,dx = \frac{1}{64}\int_1^2 xy_8(x)y_8(x)\,dx = \frac{3}{64}. \end{aligned}$$

So

Answer: $\frac{3}{64}$

7) If $\lambda = 0$ then the solutions to $y'' + \lambda y = 0$ are $y(x) = Ax + B$. We now want $y'(0) = 0$ so $B = y'(0) = 0$. Moreover we want $y'(3) = 0$ which tells us nothing new. Thus when $\lambda = 0$ we get the solution $y = 1$ with eigenvalue $\lambda = 0$.

Now if $\lambda < 0$, so we can write it $\lambda = -\nu^2$ then our solutions to the ODE are $y(x) = Ae^{\nu x} + Be^{-\nu x}$. Plugging in the boundary conditions we see

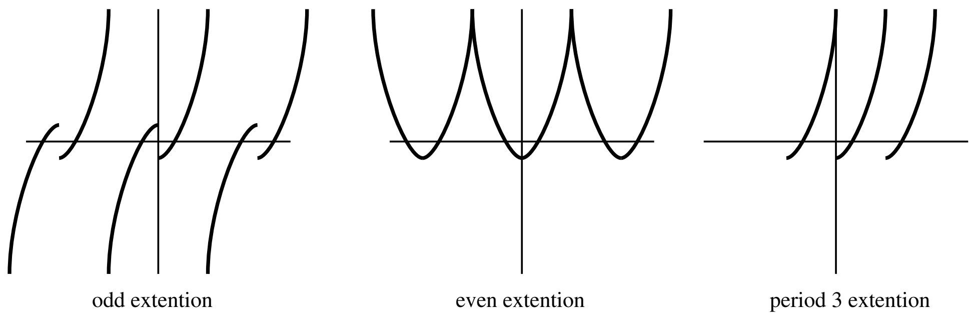
$$\nu A + \nu B = 0 \quad \text{and} \quad A\nu e^{3\nu} - B\nu e^{-3\nu} = 0.$$

One may easily check that the only A and B satisfying this are $A = 0 = B$. Thus we only get the trivial solution again.

Now if $\lambda > 0$, so we can write it $\lambda = \nu^2$ then out solutions to the ODE are $y(x) = A\cos\nu x + B\sin\nu x$. Plugging in the boundary conditions we see $B\nu = 0$ so $B = 0$ and $A\nu\sin 3\nu = 0$. To get non trivial solutions we need $3\nu = n\pi$. To get all eigenfunctions we just need to consider non-negative integers n. (Note: when $n = 0$ we get the eigen function $y = 1$ with eigenvalue $\lambda = 0$.) Thus we have

Answer: Eigenvalues $\lambda_n = \left(\frac{n\pi}{3}\right)^2$ with eigenfunctions $\cos\frac{n\pi}{3}x$, where n runs through all non-negative integers.

8)



Answer: At $x = 0$ Fourier series of f_e converges to -1.
At $x = 0$ Fourier series of f_o converges to 0.
At $x = -2$ Fourier series of f_e converges to 3.
At $x = -2$ Fourier series of f_o converges to -3.

9) We look for a power series solution $y = \sum_{n=0}^{\infty} c_n x^n$. So

$$y' = \sum_{n=1}^{\infty} c_n n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} c_n n(n-1)x^{n-2}$$

$$\begin{aligned}
(x-1)y'' &= \sum_{n=2}^{\infty} c_n n(n-1)x^{n-1} - \sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} \\
&= \sum_{n=2}^{\infty} c_n n(n-1)x^{n-1} - \sum_{n=1}^{\infty} c_{n+1}(n+1)nx^{n-1} \qquad \text{shift index} \\
&= -c_2 2x^0 + \sum_{n=2}^{\infty}(c_n n(n-1) - c_{n+1}(n+1)n)x^{n-1}
\end{aligned}$$

$$\begin{aligned}
(x-1)y'' + y' &= \left(-c_2 2x^0 + \sum_{n=2}^{\infty}(c_n n(n-1) + c_{n+1}(n+1)n)x^{n-1}\right) \\
&\quad + \left(\sum_{n=1}^{\infty} c_n n x^{n-1}\right) \\
&= -2c_2 + c_1 + \sum_{n=2}^{\infty}\left(nc_n + n(n-1)c_n - (n+1)nc_{n+1})x^{n-1}\right) = 0
\end{aligned}$$

So $c_1 - 2c_2 = 0$ and

$$c_{n+1} = \frac{n^2}{n(n+1)}c_n = \frac{n}{n+1}c_n$$

So we see

$$\begin{aligned}
c_2 &= \frac{1}{2}c_1 \\
c_3 &= \frac{2}{3}c_2 = \frac{2}{3}\frac{1}{2}c_1 = \frac{1}{3}c_1 \\
c_4 &= \frac{3}{4}(c_3) = \frac{3}{4}(\frac{1}{3}c_1) = \frac{1}{4}c_1
\end{aligned}$$

Thus we see that

$$c_n = \frac{1}{n}c_1.$$

So one solution to the equation is

$$y = c_1 \sum_{n=1}^{\infty} \frac{1}{n}x^n.$$

(You could choose $c_1 = 1$ if you like.) Notice this solution comes from choosing c_1 then the c_n's are determined for $n > 1$. Our other solution comes from choosing c_0. Thus it is $y = c_0$.

Answer:

$$y = \sum_{n=1}^{\infty} \frac{1}{n} x^n$$

$$y = 1$$