Math 241 Practice Midterm Answers

1) True False

1. True.
2. False.
3. False. For example $u(x,t) = 5$.
4. True.
5. False. The integral $\int_0^\infty e^{t^2-st}dt$ does not converge since the integrand does not go to zero as t goes to ∞.

2) Short Answer

1. $2\hat{f}(w) - \hat{g}(w)$.
2. $\frac{1}{2}$. Since solution is of the form $u(x,t) = \sum_{n=0}^\infty a_n e^{-(n\pi c)^2 t} \cos n\pi x$ all terms go to zero but $n = 0$ term so $\lim_{t\to\infty} u(x,t) = a_0$. The a_n's are the Fourier cosine series coefficients of x so $a_0 = \frac{1}{2}$.
3. Hyperbolic.
4. $s^2 F(s) - s - 1$.
5. $u(x,y) = xy$. You can solve this by guessing or by integrating.

3) The general solution to such a problem is

$$u(x,t) = \sum_{n=1}^\infty a_n e^{-\frac{9n^2\pi^2}{4}t} \sin \frac{n\pi}{2} x,$$

where the a_n are the Fourier sine series coefficients of $10 \sin 5\pi x$. Thus

$$a_n = \begin{cases} 10 & n = 10 \\ 0 & \text{otherwise.} \end{cases}$$

So our solution is

Answer:

$$u(x,t) = 10e^{-225\pi^2 t} \sin 5\pi x$$

4) Plugging $u(r,t) = R(r)T(t)$ into the equation gives

$$a^2(R''(r)T(t) + \frac{1}{r}R'(r)T(t)) = R(r)T''(t).$$

So

$$\frac{R''(r)}{R(r)} + \frac{R'(r)}{rR(r)} = \frac{T''(t)}{a^2T(t)}.$$

Since each side of this equation is a function of a different variable we must have each side being a constant, say k. Thus we have

$$rR''(r) + R'(r) = krR(r)$$

$$T''(t) = ka^2T(t)$$

So our ODE's are

Answer:

$$rR''(r) + R'(r) - krR(r) = 0, \quad T''(t) - a^2kT(t) = 0$$

5) Using the chain rule we have

$$\frac{\partial}{\partial x}f(u,v) = f_u\frac{\partial u}{\partial x} + f_v\frac{\partial v}{\partial x}$$

and $\frac{\partial u}{\partial x} = 2x$ and $\frac{\partial v}{\partial x} = 0$ so

$$\frac{\partial}{\partial x}f(u,v) = 2xf_u(u,v) = 2u^{\frac{1}{2}}f_u(u,v).$$

Similarly

$$\frac{\partial}{\partial y}f(u,v) = f_u\frac{\partial u}{\partial y} + f_v\frac{\partial v}{\partial y}$$

and $\frac{\partial u}{\partial y} = 0$ and $\frac{\partial v}{\partial y} = -\frac{1}{y^2}$ so

$$\frac{\partial}{\partial y}f(u,v) = -\frac{1}{y^2}f_v(u,v) = -v^2f_v(u,v).$$

Thus we have

Answer:

$$f_x + f_y = 2u^{\frac{1}{2}} f_u - v^2 f_v$$

6) $f(\theta) = 4\cos^2\theta = 2\cos 2\theta + 2$, so

$$b_n = 0$$

$$a_n = \begin{cases} 2 & n = 0, 2 \\ 0 & \text{otherwise} \end{cases}$$

Thus our answer is

Answer: $u(r,\theta) = 2 + 2r^2 \cos 2\theta$

7) (a) Applying the Fourier transform to both sides of the PDE we get

$$-a^2 w^2 \hat{u} = \hat{u}_{tt}.$$

The general solution to this equation is

$$\hat{u}(w,t) = c_1(w)\sin\, awt + c_2(w)\cos\, awt.$$

Fourier transforming the initial conditions gives

$$\hat{u}(w,0) = \hat{f}(w), \quad \hat{u}_t(w,t) = \hat{g}(w).$$

So $c_2(w) = \hat{f}(w)$ and $c_1(w) = \frac{\hat{g}(w)}{wa}$. Thus we have

$$\hat{u}(w,t) = \frac{\hat{g}(w)}{wa}\sin\, awt + \hat{f}(w)\cos\, awt.$$

Applying the inverse Fourier transform to this equation yields

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\hat{f}(w)\cos\, awt + \hat{g}(w)\frac{\sin\, awt}{wa}\right)e^{-iwx}\,dw$$

7) (b)

$$\begin{aligned}
u(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\cos\, awt\hat{f}(w)e^{-iwx}dw \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{iawt}+e^{-iawt}}{2}\hat{f}(w)e^{-iwx}dw \\
&= \frac{1}{2}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}(e^{iawt}+e^{-iawt})\hat{f}(w)e^{-iwx}dw\right) \\
&= \frac{1}{2}\left(\frac{1}{2\pi}(\int_{-\infty}^{\infty}\hat{f}(w)e^{-iw(x-at))}dw + \int_{-\infty}^{\infty}\hat{f}(w)e^{-iw(x+at)}dw)\right)
\end{aligned}$$

Now $\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(w)e^{-iwy}\,dw = f(y)$ so letting $y = x\pm at$ and we see $\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(w)e^{-iw(x\pm at))}\,dw = f(x\pm at)$. Thus the above equation becomes

$$u(x,t) = \frac{1}{2}(f(x-at)+f(x+at)).$$

8) The general for of the solution is

$$u(x,t) = \sum_{n=1}^{\infty} \sin\, n\pi x(a_n \cos\, an\pi t + b_n \sin\, an\pi t),$$

where the a_n's are the Fourier sine series coefficients of $x(1-x)$ and b_n's are $\frac{1}{an\pi}$ times the Fourier sine series coefficients of 0. Thus $b_n = 0$ and

$$\begin{aligned} a_n &= 2\int_0^1 x(1-x)\sin\, n\pi x\,dx \\ &= 2((x-x^2)\frac{1}{n\pi}\cos\, n\pi x - \frac{2x}{(n\pi)^2}\sin\, n\pi x - \frac{2}{(n\pi)^3}\cos\, n\pi x)|_0^1 \\ &= \frac{4}{(n\pi)^3}(1-(-1)^n), \end{aligned}$$

where the second equality is by integration by parts. Thus when n is even $a_n = 0$ and when $n = 2k-1$ we have $a_n = \frac{8}{((2k-1)\pi)^3}$. So

Answer:

$$u(x,t) = \sum_{n=1}^{\infty} \frac{8}{((2n-1)\pi)^3}\cos\left((2n-1)a\pi t\right)\sin\left((2n-1)\pi x\right)$$

9)

$$\begin{aligned} F'(t) &= \int_0^L 2u(x,t)u_t(x,t)\,dx = \int_0^L 2ku(x,t)u_{xx}(x,t)\,dx \\ &= 2ku(x,t)u_x(x,t)|_0^L - \int_0^L 2ku_x^2(x,t)\,dx \\ &= 2k(u(L,t)u_x(L,t) - u(0,t)u_x(0,t)) - \int_0^L 2ku_x^2(x,t)\,dx \\ &= -2k\int_0^L u_x^2(x,t)\,dx \le 0 \end{aligned}$$

where the equality on the second line is by integration by parts and the equality on the third line is by the boundary conditions. Thus $F(t)$ is a non increasing function of t, that means that $F(t_2) \leq F(t_1)$ for $t_2 \geq t_1$, but this is exactly what we were trying to show.

10) Using the chain rule

$$\frac{\partial}{\partial x}u(x,t) = \frac{\partial}{\partial x}f(x-ct) = f'(x-ct)\frac{\partial}{\partial x}(x-ct) = f'(x-ct).$$

Similarly

$$\frac{\partial^2}{\partial x^2}u(x,t) = f''(x-ct),$$

and

$$\frac{\partial^2}{\partial t^2}u(x,t) = c^2 f''(x-ct).$$

So the PDE becomes

$$c^2 f''(x-ct) = af''(x-ct) - bf(x-ct).$$

Tidying up a bit gives

Answer:

$$f''(x-ct) = \frac{-b}{c^2-a}f(x-ct)$$

Now thinking of f as a function of y we solve $f'' = \frac{-b}{c^2-a}f$. Note $\frac{-b}{c^2-a} < 0$ so the solution is

$$f(y) = c_1 \sin\sqrt{\frac{b}{c^2-a}}y + c_2 \cos\sqrt{\frac{b}{c^2-a}}y.$$

So our solution to the PDE is

Answer:

$$u(x,t) = c_1 \sin\sqrt{\frac{b}{c^2-a}}(x-ct) + c_2 \cos\sqrt{\frac{b}{c^2-a}}(x-ct)$$