Math 241 Practice Midterm

Name: ____________________________________

Student ID: ________________________________

Signature: _________________________________

Instructions: Print your name and student ID number and sign your signature to indicate that you accept the honor code. You may use **one page** of notes on this test. You may not use any other notes, books, calculators or computers. When a box is provided for your answer you must write your answer (and nothing else) in the box to receive full credit for the problem. Even if the correct answer appears somewhere else on the page, you will not receive full credit. Moreover, you must also **show the work** you did to arrive at the answer to receive full credit.You have 1.5 hours to answer all the questions. *Good Luck*

Question	Max Point	Score
1	5	
2	5	
3	5	
4	4	
5	3	
6	3	
7	6	
8	5	
9	4	
10	2(+1)	
Total	43(+1)	

1) True or False: (1 point each) Circle **T** for True and **F** for False.

1. The Laplace transform of a function $f(t), t \geq 0$, is defined by

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt.$$

T **F**

2. A partial differential equation without boundary and initial conditions always has a unique solution.

T **F**

3. For any fixed x, the temperature $u(x,t)$ of a bar with insulated ends always tends to zero, as $t \to \infty$.

T **F**

4. The functions $v(x,y) = xy$ and $w(x,y) = x^2 - y^2$ solve the partial differential equation $u_{xx} + u_{yy} = 0$.

T **F**

5. The Laplace transform of the function $f(x) = e^{x^2}$ is well-defined.

T **F**

2)Short Answer: (1 point each)

1. If $f(x)$ and $g(x)$ have Fourier transforms $\hat{f}(w)$ and $\hat{g}(w)$ then what is the Fourier transform of $2f(x) - g(x)$?

Answer:

2. Let $u(x,t), 0 \leq x \leq 1, t \geq 0$, satisfy the one-dimensional heat equation with boundary conditions $u_x(0,t) = 0, u_x(1,t) = 0$, and initial condition $u(x,0) = x, 0 < x < 1$. For all $0 < x < 1$, what does $u(x,t)$ tend to as $t \to \infty$?

Answer:

3. Is the equation $u_{xx} - 3u_{xy} + u_{yy} + 7u_x + 8u_y + u = e^{xy}$ elliptic, hyperbolic or parabolic?

Answer:

4. If $f(t)$ has Laplace transform $F(s)$ and $f(0) = f'(0) = 1$ then what is the Laplace transform of $f''(t)$?

Answer:

5. Find a solution $u(x,y)$ to the partial differential equation $u_{xy} = 1$.

Answer:

3) (5 points) Find a solution to $u_t = 9u_{xx}$ which satisfies the boundary conditions $u(0,t) = 0 = u(2,t)$ and the initial conditions $u(x,0) = 10\sin\, 5\pi x$.

Answer:

4) (4 points) Consider the PDE

$$a^2\left(u_{rr}+\frac{1}{r}u_r\right)=u_{tt}.$$

Looking for a solution of the form $u(r,t) = R(r)T(t)$ use the separation of variables technique to find the ODE's that $R(r)$ and $T(t)$ must satisfy. (Do NOT try to solve these equations.)

Answer:

5) (3 points) Consider the function $f(x,y)$. Express $f_x + f_y$ in terms of the new coordinates u and v where

$$x = u^{\frac{1}{2}}, \quad y = \frac{1}{v}$$

Answer:

6) (3 points) Let r and θ denote polar coordinates on the plane. Then solution $u(r, \theta)$ to $\nabla^2 u = 0$ in the disk of radius 1 which satisfies the boundary conditions $u(1, \theta) = f(\theta), 0 \leq \theta < 2\pi$, can be written as

$$u(r, \theta) = a_0 + \sum_{n=1}^{\infty} (a_n r^n \cos n\theta + b_n r^n \sin n\theta).$$

Assume that $f(\theta) = 4\cos^2\theta$. Find $u(r, \theta)$.

Answer:

7) Consider the problem

$$a^2 u_{xx} = u_{tt}, \quad -\infty < x < \infty, t > 0$$

$$u(x,0) = f(x), \quad u_t(x,0) = g(x).$$

(a) (3 points) Using the Fourier transform show that

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\hat{f}(w) \cos awt + \hat{g}(w) \frac{\sin awt}{wa} \right) e^{-iwx} \, dw.$$

(b) (3 points) If $g(x) = 0$ show the solution from above becomes $u(x,t) = \frac{1}{2}[f(x+at) + f(x-at)]$.

8) (5 points) Solve

$$u_{tt} = a^2 u_{xx}, \quad 0 \leq x \leq 1$$

$$u(0,t) = 0, \quad u(1,t) = 0$$

$$u(x,0) = x(1-x), \quad u_t(x,0) = 0$$

Answer:

9) (4 points) Let $u(x,t)$ solve $u_t = ku_{xx}$, on $0 \leq x \leq L$ and $t \geq 0$, where k is a positive constant. Also suppose $u_x(0,t) = 0$ and $u_x(L,t) = 0$. With out using the explicit form of the solution to such a problem derived in class show that for all t_1, t_2 such that $t_2 \geq t_1 \geq 0$,

$$\int_0^L (u(x,t_2))^2 \, dx \leq \int_0^L (u(x,t_1))^2 \, dx.$$

HINT: Consider $F(t) = \int_0^L (u(x,t))^2 \, dx$.

10) (2 points) Consider the equation $u_{tt} = au_{xx} - bu$, where b, a are constants. If we look for a solution to this equation of the form $u(x,t) = f(x - ct)$ then derive an ODE that f must satisfy.

Answer:

Bonus Problem(+1 point) Find all such solutions to the equation when $c \geq \sqrt{a}$.

Answer: