CHARACTERS AND q-SERIES IN $\mathbb{Q}(\sqrt{2})$

DANIEL CORSON, DAVID FAVERO, KATE LIESINGER, AND SARAH ZUBAIRY

Abstract. In 1988, G. Andrews, F. Dyson, and D. Hickerson related the arithmetic of $\mathbb{Q}(\sqrt{6})$ to certain q-series. We have found q-series that relate in a similar way to $\mathbb{Q}(\sqrt{2})$. In addition to proving analogous results, including an explicit formula for a partition function, we also obtain a generating function for the values of a particular L-function.

1. Introduction and Statement of Results

In [3], G. Andrews, F. Dyson, and D. Hickerson study the relationship between the arithmetic of $\mathbb{Q}(\sqrt{6})$ and certain partition functions. This connection allows them to prove new results about combinatorial objects by taking a non-combinatorial perspective. They were interested in the following q-series:

$$R(q) = 1 + \sum_{n=1}^{\infty} \frac{q^{n(n+1)/2}}{(1+q)(1+q^2)\cdots(1+q^n)} = 1 + q - q^2 + 2q^3 - 2q^4 + \cdots$$

$$L(q) = 2\sum_{n=1}^{\infty} \frac{(-1)^n q^{n^2}}{(1-q)(1-q^3)...(1-q^{2n-1})} = -2q - 2q^2 - 2q^3 + 2q^7 + \cdots$$

They showed that the coefficients of $R(q)$ and $L(q)$ are determined by the coefficients of a certain Hecke L-function associated with the quadratic field $\mathbb{Q}(\sqrt{6})$. Using the arithmetic of $\mathbb{Q}(\sqrt{6})$, the combinatorics of q-series, and basic hypergeometric series, they proved a number of results about the coefficients of

$$qR(q^{24}) - \frac{1}{q}L(q^{24}),$$

including its multiplicativity and lacunarity. They also show that the coefficients attain every integer infinitely often. Examples of q-series with these properties are rare and surprising. In the words of F. Dyson [6],

> *This pair of functions $R(q)$ and $L(q)$ is today an isolated curiosity. But I am convinced that, like so many other beautiful things in Ramanujan's garden, it will turn out to be a special case of a broader mathematical structure. There probably exist other sets of two or more functions with coefficients related by cross-multiplicativity, satisfying identities similar to those which Ramanujan discovered for his $R(q)$. I have a hunch that such sets of cross-multiplicative functions will form a structure within which the mock theta-functions will also find a place. But this hunch is not backed up by any solid evidence. I leave it to the ladies and gentlemen of the audience to find the connections if they exist.*

In this paper we find q-series analogous to $R(q)$ and $L(q)$, associated in a similar way to $\mathbb{Q}(\sqrt{2})$. We relate a sum of these basic hypergeometric series with a Hecke L-function, using the machinery of Bailey pairs. We prove analogous combinatorial results to those in [3]; using the arithmetic of $\mathbb{Q}(\sqrt{2})$, we establish combinatorial properties of a certain partition function. In addition, we find a generating function for values of the associated L-function.

Throughout the paper we employ the standard notation

$$(a)_n := (a;q)_n := \prod_{k=0}^{n-1}(1 - aq^k).$$

2000 *Mathematics Subject Classification.* 11P81, 11M41.

The authors thank the following for their generous support: the John S. Guggernheim Foundation, David and Lucile Packard Foundation, Alfred P. Sloan Foundation, National Science Foundation, and University of Wisconsin at Madison.

Let $O_K = \mathbb{Z}[\sqrt{2}]$ be the ring of integers of $K = \mathbb{Q}(\sqrt{2})$. In O_K define the norm of any ideal $\mathfrak{a} = (x + y\sqrt{2})$ as $N(\mathfrak{a}) := |x^2 - 2y^2|$.

Define $W_1(q)$ and $W_2(q)$ by the q-series

$$W_1(q) := \sum_{n\geq 0} \frac{(q)_n(-1)^n q^{\binom{n+1}{2}}}{(-q)_n} = 1 - q + 2q^2 - q^3 - 2q^5 + 3q^6 + \cdots, \tag{1.1}$$

$$W_2(q) := \sum_{n\geq 1} \frac{(-1;q^2)_n(-q)^n}{(q;q^2)_n} = -2q - 2q^3 + 2q^4 + 2q^6 + 2q^8 - 2q^9 + \cdots. \tag{1.2}$$

Let χ be the character

$$\chi(\mathfrak{a}) := \begin{cases} 1 & N(\mathfrak{a}) \equiv \pm 1 \bmod 16 \\ -1 & N(\mathfrak{a}) \equiv \pm 7 \bmod 16 \\ 0 & \text{otherwise,} \end{cases} \tag{1.3}$$

and define $a(n)$ for any positive integer n by

$$a(n) := \sum_{\substack{\mathfrak{a}\subset O_K \\ N(\mathfrak{a})=n}} \chi(\mathfrak{a}). \tag{1.4}$$

Theorem 1. *We have*

$$qW_1(q^8) + \frac{1}{q}W_2(q^8) = \sum_{n\geq 0} a(n)q^n. \tag{1.5}$$

Remark. The $a(n)$'s are constructed such that the following holds ($\Re(s) > 1$):

$$L(\chi, s) := \sum_{\mathfrak{a}\subset O_K} \frac{\chi(\mathfrak{a})}{N(\mathfrak{a})^s} = \sum_{n\geq 1} \frac{a(n)}{n^s}. \tag{1.6}$$

In particular, $L(\chi, s)$ is a standard Hecke L-function which is well-known to have an analytic continuation to $\mathbb{C}$ [2].

Corollary 2. *The following identity is true:*

$$qW_1(-q^8) + \frac{1}{q}W_2(-q^8) = \sum_{\substack{n\geq 1 \\ n \text{ odd}}} b(n)q^n,$$

where the $b(n)$'s are defined by

$$b(n) := \sum_{\substack{n \text{ odd} \\ \mathfrak{a}\subset O_K \\ N(\mathfrak{a})=n}} 1.$$

Remark. The $b(n)$'s are constructed such that the following holds ($\Re(s) > 1$):

$$\zeta_K^*(s) := \sum_{\substack{\mathfrak{a}\subset O_K \\ N(\mathfrak{a}) \text{ odd}}} \frac{1}{N(\mathfrak{a})^s} = \sum_{\substack{n\geq 1 \\ n \text{ odd}}} \frac{b(n)}{n^s}.$$

Notice $\zeta_K^*(s)$ is essentially the usual Dedekind ζ-function, but the only difference is the omission of the Euler factor corresponding to the prime ideal above 2. Here $\zeta_K^*(s)$ has an analytic continuation to $\mathbb{C}$ with the exception of a simple pole at $s = 1$ (for example, see [8]).

Consider the q-series identity in (1.5) with $q = e^{-t}$. This gives a well-defined t-series, since the substitution of $e^{-t} = \sum_{n=0}^{\infty} \frac{(-t)^n}{n!}$ into (1.1) amounts to performing formal operations (addition, multiplication, and taking positive integral powers) of power series.

Theorem 3. *The following is a generating function for L-values.*

$$e^{-t}W_1(e^{-8t}) - e^t \sum_{n\geq 0} \frac{(e^{-8t};e^{-16t})_n}{(-e^{-16t};e^{-16t})_n} = \sum_{n\geq 0} L(\chi, -n)\frac{(-1)^{n+1}t^n}{n!} = -10t - \frac{7949}{3}t^3 - \frac{26765521}{12}t^5 - \cdots$$

Theorem 1 is proven in two steps. In Section 2, using the theory of Bailey pairs, we find alternate expressions for $W_1(q)$ and $W_2(q)$, and in Section 3 we prove our theorem by revealing the connection to $\mathbb{Q}(\sqrt{2})$ of these other representations. In Section 4 we prove Corollary 2. In Section 5 we find an explicit formula for the coefficients of our q-series, and provide combinatorial results. In Section 6 we establish the generating function for L-values.

2. Hecke Identities

Here we employ the theory of Bailey pairs to obtain alternate q-series expressions for $W_1(q)$ and $W_2(q)$.

Definition 2.1. *Two sequences α_n and β_n, form a Bailey pair relative to a if for all $n \geq 0$*

$$\beta_n = \sum_{r=0}^{n} \frac{\alpha_r}{(q)_{n-r}(aq)_{n+r}}.$$

Theorem 2.2. *(Bailey's Lemma) If α_n and β_n form a Bailey pair relative to a, then*

$$\sum_{n\geq 0} \frac{(\rho_1)_n(\rho_2)_n(aq/\rho_1\rho_2)^n}{(aq/\rho_1)_n(aq/\rho_2)_n}\alpha_n = \frac{(aq)_\infty(aq/\rho_1\rho_2)_\infty}{(aq/\rho_1)_\infty(aq/\rho_2)_\infty}\sum_{n\geq 0}(\rho_1)_n(\rho_2)_n(aq/\rho_1\rho_2)^n\beta_n,$$

provided that both sums converge absolutely.

A proof can be found in [1].

Theorem 2.3. *The following identity is true:*

$$W_1(q) = \sum_{\substack{n\geq 0\\ |j|\leq n}} (-1)^{n+j} q^{2n^2+n-j^2}(1-q^{2n+1}). \tag{2.1}$$

Proof. Recall that

$$W_1(q) := \sum_{n\geq 0} \frac{(q)_n(-1)^n q^{\binom{n+1}{2}}}{(-q)_n}.$$

In Bailey's Lemma, let $\rho_1 \to \infty$, $\rho_2 = q$ and $a = q$. Note that when $\rho_1 \to \infty$ then $(\rho_1)_n\left(\frac{1}{\rho_1}\right)^n \to (-1)^n q^{\binom{n}{2}}$. This yields

$$\sum_{n\geq 0}(-1)^n q^{\binom{n+1}{2}}\alpha_n = \frac{1}{1-q}\sum_{n\geq 0}(-1)^n q^{\binom{n}{2}}(q)_n q^n \beta_n. \tag{2.2}$$

By [4], the following form a Bailey pair relative to $a = q$

$$\alpha_n = \frac{q^{(3n^2+n)/2}(1-q^{2n+1})}{1-q}\sum_{j=-n}^{n}(-1)^j q^{-j^2} \quad \text{and} \quad \beta_n = \frac{1}{(-q)_n}.$$

Substitution into (2.2) gives the result. □

Theorem 2.4. *The following identity is true:*

$$W_2(q) = \sum_{\substack{n\geq 1\\ -n\leq j\leq n-1}} (-1)^n q^{n(2n-1)-(j^2-j)}(1+q^{2n}). \tag{2.3}$$

Proof. Recall that

$$W_2(q) := \sum_{n\geq 1} \frac{(-1;q^2)_n(-q)^n}{(q;q^2)_n}.$$

Make the substitution $q \to \sqrt{q}$ and shift the sums via $n \to n+1$. The left hand side becomes

$$\sum_{n\geq 0} \frac{(-1)_{n+1}(-\sqrt{q})^{n+1}}{(\sqrt{q})_{n+1}} = \frac{-2\sqrt{q}}{1-\sqrt{q}}\sum_{n\geq 0}\frac{(-q)_n(-\sqrt{q})^n}{(q^{3/2})_n}.$$

The right hand side becomes

$$-\sum_{n\geq 0}(-1)^n q^{(2n^2+3n+1)/2}(1+q^{n+1})(\sum_{j=0}^{n} q^{-j(j+1)/2}+\sum_{j=-n-1}^{-1} q^{-j(j+1)/2}).$$

Flip the last sum by taking $i=-(j+1)$ to get

$$-\sum_{n\geq 0}(-1)^n q^{(2n^2+3n+1)/2}(1+q^{n+1})(\sum_{j=0}^{n} q^{-j(j+1)/2}+\sum_{i=0}^{n} q^{-i(i+1)/2}),$$

and then combine sums

$$-\sum_{n\geq 0}(-1)^n q^{(2n^2+3n+1)/2}(1+q^{n+1})(2\sum_{j=0}^{n} q^{-j(j+1)/2}).$$

It remains to show

$$-2\sqrt{q}\sum_{n\geq 0}(-1)^n q^{n^2+3n/2}(1+q^{n+1})(\sum_{j=0}^{n} q^{-j(j+1)/2})=\frac{-2\sqrt{q}}{1-\sqrt{q}}\sum_{n\geq 0}\frac{(-q)_n(-\sqrt{q})^n}{(q^{3/2})_n}.$$

The following is a Bailey pair relative to $a=q^2$:

$$\alpha_n=\frac{q^{n^2+n}(1-q^{2n+2})}{(1-q^2)}\sum_{j=0}^{n} q^{-j(j+1)/2} \quad\text{and}\quad \beta_n=\frac{(-q)_n}{(q)_n(-q^{3/2})_n(q^{3/2})_n},$$

as can be seen by taking $b=-q^{1/2}$ and $c=q^{1/2}$ in Theorem 2.2 in [4]. Apply Bailey's lemma to this pair, choosing $\rho_1=-q^{3/2}$ and $\rho_2=q$, to obtain

$$\frac{1}{(1+q)}\sum_{n\geq 0}(-1)^n q^{n^2+3n/2}(1+q^{n+1})\sum_{j=0}^{n} q^{-j(j+1)/2}=\frac{(1+\sqrt{q})}{(1-q^2)}\sum_{n\geq 0}\frac{(-\sqrt{q})^n(-q)_n}{(q^{3/2})_n}.$$

Multiplying both sides by $-2\sqrt{q}(1+q)$ and simplifying yields the identity. □

3. Proof of Theorem 1

Theorem 1 will be an immediate consequence of (2.1) and (2.3) once we know that the only ideals $\mathfrak{a}$ with $\chi(\mathfrak{a})\neq 0$ have $N(\mathfrak{a})\equiv \pm 1 \bmod 8$. The following lemma establishes that.

Lemma 3.1. *There are no ideals of norm* ± 3 *mod* 8 *in* O_K*.*

Proof. Consider any ideal $\mathfrak{a}=(x+y\sqrt{2})$ with $x^2-2y^2=8n+3$ for some $n\in\mathbb{Z}$. Look mod 2 to see x must be odd, $x=2k+1$. Then $4k^2+4k+1-2y^2=8n+3$, so $2k^2+2k-y^2=4n+1$. Looking mod 2 again shows y must also be odd, $y=2m+1$. Then $2k^2+2k-4m^2-4m-1=4n+1$, so $k(k+1)-2m^2-2m=2n+1$. If we look mod 2 again, we have that $k(k+1)$ is odd. But that is impossible. The proof for $N(\mathfrak{a})=-3 \bmod 8$ is similar. □

Theorem 3.2. *The following identity is true:*

$$qW_1(q^8)=\sum_{\substack{n\geq 0\\ n\equiv 1 \bmod 8}} a(n)q^n. \tag{3.1}$$

Proof. The fundamental solution of $x^2-2y^2=1$ (the solution with x and y minimal positive) is (3,2). From [4] (Lemma 3, pg 396), we know that we choose a unique representative of each ideal $\mathfrak{a}=(x+y\sqrt{2})$ in O_K by restricting $x\geq 0$ and $-\frac{2}{3+1}x<y\leq\frac{2}{3+1}x$.

Suppose $x^2-2y^2=8m+1$. Looking mod 2, we see x is odd. Write $x=2k+1$. The inequalities become $k\geq 0$ and $|y|\leq k$. Note that since $N(\mathfrak{a})\equiv 1 \bmod 8$, from (1.3) we can say $\chi(\mathfrak{a})=(-1)^{\frac{N(\mathfrak{a})-1}{8}}$. This gives the following:

$$\sum_{\substack{n\geq 0\\ n\equiv 1 \bmod 8}} a(n)q^n=\sum_{\substack{k\geq 0\\ |y|\leq k}}(-1)^{\frac{k^2+k}{2}-\frac{y^2}{4}}q^{(2k+1)^2-2y^2}.$$

Now we split into two sums, corresponding to the cases $k = 2n+1$ and $k = 2n$. Since y must always be even, take $y = 2j$.

$$\sum_{\substack{n\geq 0\\ |j|\leq n}} (-1)^{n+j+1} q^{(4n+3)^2-8j^2} + \sum_{\substack{n\geq 0\\ |j|\leq n}} (-1)^{n+j} q^{(4n+1)^2-8j^2}$$

Combining these two sums we get the result:

$$\sum_{\substack{n\geq 0\\ |j|\leq n}} (-1)^{n+j} q^{(4n+1)^2-8j^2}(1-q^{8(2n+1)}).$$

□

Theorem 3.3. *The following identity is true:*

$$\frac{1}{q}W_2(q^8) = \sum_{\substack{n\geq 0\\ n\equiv -1 \bmod 8}} a(n)q^n. \tag{3.2}$$

Proof. Suppose $x^2 - 2y^2 = 8m-1$. From (1.3), $\chi(\mathfrak{a}) = (-1)^{\frac{N(\mathfrak{a})+1}{8}}$. Again, x must be odd, $x = 2k+1$, and now y is also odd, $y = 2j+1$. To ensure a unique representative of each ideal, we use the inequalities above, $k \geq 0$ and $|y| \leq k$. Consider the two sums, $k = 2n+1$ and $k = 2n$.

$$\sum_{\substack{n\geq 0\\ n\equiv -1 \bmod 8}} a(n)q^n = \sum_{\substack{n\geq 0\\ -n-1\leq j\leq n}} (-1)^{n+1} q^{(4n+3)^2-2(2j+1)^2} + \sum_{\substack{n\geq 0\\ -n\leq j\leq n-1}} (-1)^{n} q^{(4n+1)^2-2(2j+1)^2}.$$

Shifting the first sum and combining them we get the result,

$$\sum_{\substack{n\geq 1\\ -n\leq j\leq n-1}} (-1)^{n} q^{(4n-1)^2-2(2j+1)^2}(1+q^{16n}).$$

□

4. Proof of Corollary 2

Corollary 2 gives the result of Theorem 1 on the trivial character

$$|\chi|(\mathfrak{a}) = \begin{cases} 1 & N(\mathfrak{a}) \equiv \pm 1, \pm 7 \bmod 16 \\ 0 & \text{otherwise,} \end{cases}$$

with the particularly simple associated L-function $\zeta_K^*(s)$. Instead of repeating the methods used to prove Theorem 1, however, we can use Theorem 1 more directly.

Proof of Corollary 2. Let $\gamma := e^{2\pi i/16}$, be a primitive 16^{th} root of unity. Substitute $q \to \gamma q$ in (3.1):

$$\gamma q W_1((\gamma q)^8) = \sum_{\substack{\mathfrak{a}\subset O_K\\ N(\mathfrak{a})\equiv 1 \bmod 8}} \chi(\mathfrak{a})(\gamma q)^{N(\mathfrak{a})}.$$

Dividing through by γ shows

$$qW_1(-q^8) = \sum_{\substack{\mathfrak{a}\subset O_K\\ N(\mathfrak{a})\equiv 1 \bmod 8}} \chi(\mathfrak{a})\gamma^{N(\mathfrak{a})-1}q^{N(\mathfrak{a})}.$$

¿From (1.3), $\chi(\mathfrak{a}) = (-1)^{\frac{N(\mathfrak{a})-1}{8}}$ when $N(\mathfrak{a}) \equiv 1 (\bmod 8)$, thus

$$qW_1(-q^8) = \sum_{\substack{\mathfrak{a}\subset O_K\\ N(\mathfrak{a})\equiv 1 \bmod 8}} q^{N(\mathfrak{a})} = \sum_{n\equiv 1 \bmod 8} b(n)q^n.$$

Substitute $q \to \gamma q$ in (3.2),

$$\frac{1}{\gamma q}W_2(\gamma q) = \sum_{\substack{n\geq 0\\ n\equiv -1 \bmod 8}} a(n)(\gamma q)^n.$$

Multiplying through by γ gives

$$qW_2(-q^8) = \sum_{\substack{\mathfrak{a}\subset O_K \\ N(\mathfrak{a})\equiv -1 \bmod 8}} \chi(\mathfrak{a})\gamma^{N(\mathfrak{a})+1} q^{N(\mathfrak{a})}.$$

Similarly, $\chi(\mathfrak{a}) = (-1)^{\frac{N(\mathfrak{a})+1}{8}}$ when $N(\mathfrak{a}) \equiv -1 \bmod 8$, thus

$$qW_2(-q^8) = \sum_{\substack{\mathfrak{a}\subset O_K \\ N(\mathfrak{a})\equiv -1 \bmod 8}} q^{N(\mathfrak{a})} = \sum_{n\equiv -1 \bmod 8} b(n)q^n.$$

Since there are no ideals of norm $\pm 3 \bmod 8$ in O_K, the result follows. □

5. Combinatorial Interpretation

The q-series $W_1(q)$ has interesting combinatorial properties. It is related the Rogers-Ramanujan type identity ([10], eq. 8):

$$\sum_{n=0}^{\infty} \frac{(-q)_n q^{\binom{n+1}{2}}}{(q)_n} = \frac{(-q^2;q^2)_\infty}{(q;q^2)_\infty}.$$

It is also a generating function for certain types of partitions. If

$$W_1(q) := \sum_{n\geq 0} \frac{(q)_n(-1)^n q^{\binom{n+1}{2}}}{(-q)_n} = \sum_{n\geq 0} A(n)q^n,$$

then $A(n)$ counts the number of colored partitions of n into *quasi-distinct* parts where the largest yellow part is less than or equal to the number of purple parts, weighted by $(-1)^{P+Y}$ where P is the largest purple part and Y is the number of yellow parts. Here, *quasi-distinct* means no two parts can have both the same value and color, but there may be two parts of the same value and different colors. Notice from (3.1) that $A(n) = a(8n+1)$.

Example When $n = 4$, the colored partitions of this type are 4 and $3+1'$ with weight 1, and $3+1$ and $2+1+1'$ with weight -1 (unprimed numbers are purple parts, primed numbers are yellow parts). So $A(4) = 0$. There are no ideals of norm 33 in O_K, so $a(8\cdot 4+1) = 0$ as well.

Example When $n = 5$, the colored partitions of this type are $4+1$ and $3+1+1'$ with weight 1; and 5, $4+1'$, $3+2$, and $2+2'+1$ with weight -1. So $A(5) = -2$. The ideals of norm 41 in O_K are $(7+2\sqrt{2})$ and $(7-2\sqrt{2})$, and since χ is -1 for both these ideals as $41 \equiv -7 \bmod 16$, $a(41) = -2$.

The following two results establish a general formula for the $a(n)$'s, which we use to study $A(n)$.

Lemma 5.1. *The $a(n)$'s are multiplicative. That is, if* $\gcd(n,m) = 1$ *then* $a(nm) = a(n)a(m)$.

Proof. Recall the definition of $a(n)$

$$a(n) := \sum_{\substack{\mathfrak{a}\subset O_K \\ N(\mathfrak{a})=n}} \chi(\mathfrak{a}).$$

Suppose we have an ideal $\mathfrak{a}$ with $N(\mathfrak{a}) = nm$. It is well known that $\mathbb{Z}[\sqrt{2}]$ is a UFD, so factor the ideal $\mathfrak{a} = \mathfrak{p}_1\mathfrak{p}_2\cdots\mathfrak{p}_k$. Then $nm = N(\mathfrak{p}_1)N(\mathfrak{p}_2)\cdots N(\mathfrak{p}_k)$, since the norm is multiplicative. Because n and m are coprime, there must be a (set theoretic) partition $\{n_1,\cdots,n_r\}\cup\{m_1,\cdots,m_s\} = \{1,\cdots,k\}$ such that $n = N(\mathfrak{p}_{n_1})N(\mathfrak{p}_{n_2})\cdots N(\mathfrak{p}_{n_r})$ and $m = N(\mathfrak{p}_{m_1})N(\mathfrak{p}_{m_2})\cdots N(\mathfrak{p}_{m_s})$. Let $\mathfrak{b} = \mathfrak{p}_{n_1}\mathfrak{p}_{n_2}\cdots\mathfrak{p}_{n_r}$ and $\mathfrak{c} = \mathfrak{p}_{m_1}\mathfrak{p}_{m_2}\cdots\mathfrak{p}_{m_s}$. Then $\mathfrak{a} = \mathfrak{b}\mathfrak{c}$ and $N(\mathfrak{b}) = n$ and $N(\mathfrak{c}) = m$. So

$$a(nm) = \sum_{\substack{\mathfrak{a}\subset O_K \\ N(\mathfrak{a})=nm}} \chi(\mathfrak{a}) = \sum_{\substack{\mathfrak{b},\mathfrak{c}\subset O_K \\ N(\mathfrak{b})=n \\ N(\mathfrak{c})=m}} \chi(\mathfrak{b})\chi(\mathfrak{c}) = \left(\sum_{\substack{\mathfrak{b}\subset O_K \\ N(\mathfrak{b})=n}} \chi(\mathfrak{b})\right)\left(\sum_{\substack{\mathfrak{c}\subset O_K \\ N(\mathfrak{c})=m}} \chi(\mathfrak{c})\right) = a(n)a(m).$$

□

Theorem 5.2. *If p is prime and $e \geq 0$, then*

$$a(p^e) = \begin{cases} (e+1) & \text{if } a(p) = 2 \text{ and } p \equiv \pm 1 \bmod 8 \\ (-1)^e(e+1) & \text{if } a(p) = -2 \text{ and } p \equiv \pm 1 \bmod 8 \\ (-1)^{e/2} & \text{if } e \text{ is even and } p \equiv \pm 3 \bmod 8 \\ 0 & \text{if } p = 2 \text{ or } e \text{ is odd and } p \equiv \pm 3 \bmod 8. \end{cases} \tag{5.1}$$

Proof. Since $\chi(\mathfrak{a}) = 0$ when $N(\mathfrak{a})$ is even, we have $a(2^e) = 0$. For p an odd prime, 2 is a quadratic residue mod p if and only if $p \equiv \pm 1 \bmod 8$, and it is exactly in this case that (p) splits in O_K.

In the splitting case, let p factor as $\alpha\beta$. Since α and β are the only elements of norm p, the elements of norm p^e are exactly the $e+1$ elements of the form $\alpha^k\beta^l$ where $k+l = e$ and $\chi(\alpha^k\beta^l) = \chi(\alpha)^k\chi(\beta)^l$. Since α and β are conjugate, and hence have the same norm, $\chi(\alpha) = \chi(\beta)$, and so $\chi(\alpha^k\beta^l) = \chi(\alpha)^e$. When $a(p) = 2$, $\chi(\alpha) = 1$, and when $a(p) = -2$, $\chi(\alpha) = -1$. There are no other possibilities for $a(p)$ since $\chi(\alpha) = \chi(\beta)$. This gives the first two cases.

Now suppose $p \equiv \pm 3 \bmod 8$. There are no ideals of norm p^e when e is odd by Lemma 3.1, because $p^e \equiv \pm 3 \bmod 8$.

When e is even, the only ideal of norm p^e is $(p^{e/2})$, with factorization $(p)^{e/2}$, since p does not split. Here (p) is the unique ideal of norm p^2 and $\chi(p) = -1$; since $p^2 \equiv 9 \bmod 16$ when $p \equiv \pm 3, \pm 5 \bmod 16$. Thus $\chi(p^{e/2}) = (-1)^{e/2}$. □

Remark. It is well-known that in a number field with degree greater than 1 over $\mathbb{Q}$, the number of positive integers that are norms of ideals has density 0 [9]. This immediately gives that $A(n)$ is almost always 0.

Corollary 5.3. *$A(n)$ hits every integer infinitely many times.*

Proof. Given any integer $k \geq 2$ consider any $p \equiv 1 \bmod 8$. Then $p^{k-1} \equiv 1 \bmod 8$ and $9p^{k-1} \equiv 1 \bmod 8$. Let $n = (p^{k-1}-1)/8$ and $m = 9(p^{k-1}-1)/8$. If $a(p) = 2$ then $A(n) = a(8n+1) = a(p^{k-1}) = k$ and $A(m) = a(8m+1) = a(9p^{k-1}) = -k$. If $a(p) = -2$ then $A(n) = a(8n+1) = a(p^{k-1}) = (-1)^{k+1}k$ and $A(m) = a(8m+1) = a(9p^{k-1}) = (-1)^k k$. Since there are infinitely many primes, $p \equiv 1 \bmod 8$, there must be infinitely many p in at least one of these two cases. Thus $A(n)$ hits $\pm k$ infinitely many times.

For the $k = 1$ case, consider any $p \equiv \pm 3 \bmod 8$. For any even e, $p^e \equiv 1 \bmod 8$. Let $n = (p^e - 1)/8$, then $A(n) = a(8n+1) = a(p^e) = (-1)^{e/2}$. So $A(n)$ hits ± 1 infinitely many times. □

6. Proof of Theorem 3

We prove the generating function for L-values (Theorem 3) in two steps. Theorem 6.1 is a corollary to Theorem 1 which proves the existence and gives an explicit form of the asymptotic expansion of $\sum_{n=1}^{\infty} a(n)e^{-nt}$. Then, independent of Theorem 1, we prove that an asymptotic expansion of $\sum_{n=1}^{\infty} a(n)e^{-nt}$ is in fact a generating function for L-values.

Theorem 6.1. *As $t \searrow 0$ we have*

$$\sum_{n=1}^{\infty} a(n)e^{-nt} \sim e^{-t}W_1(e^{-8t}) - e^t \sum_{n\geq 0} \frac{(e^{-8t}; e^{-16t})_n}{(-e^{-16t}; e^{-16t})_n}.$$

Proof. Recall (Theorem 1) that

$$\sum_{n\geq 1} a(n)q^n = qW_1(q^8) + \frac{1}{q}W_2(q^8). \tag{6.1}$$

We will make the specialization $q = e^{-t}$ and then demonstrate convergence of the resulting t-series. In the first term

$$W_1(e^{-8t}) = \sum_{n\geq 0} \frac{e^{-8tn(n+1)/2}(-1)^n(e^{-8t}; e^{-8t})_n}{(-e^{-8t}; e^{-8t})_n},$$

is a convergent t-series since $(e^{-8t}; e^{-8t})_n \to 0$. For the second term, it can be seen that $W_2(e^{-8t})$ is asymptotically, as $t \searrow 0$, equal to the following convergent t-series when we let $t = q, q = q^2$, and $a = -q^2$ in Theorem 1 of [5]:

$$W_2(e^{-8t}) \sim \sum_{n\geq 0} \left(\frac{(e^{-8t}; e^{-16t})_\infty}{(-e^{-16t}; e^{-16t})_\infty} - \frac{(e^{-8t}; e^{-16t})_n}{(-e^{-16t}; e^{-16t})_n} \right).$$

The first term in the sum goes to 0 as $t \searrow 0$. The result is now just a matter of substituting $q = e^{-t}$ in (6.1), and applying the above observations. □

We are now ready to prove Theorem 3.

Proof of Theorem 3. The proof is analogous to the proof of Proposition 3.1 in [7]. Note that $L(\chi, s)$ has an analytic continuation to $\mathbb{C}$. Suppose the asymptotic expansion as $t \searrow 0$ is given by

$$\sum_{n\geq 1} a(n)e^{-nt} \sim \sum_{n\geq 0} c(n)t^n. \tag{6.2}$$

Consider the following integral (assume $\Re(s) > 1$):

$$\int_0^\infty \left(\sum_{n\geq 1} a(n)e^{-nt}\right) t^{s-1}dt = \sum_{n\geq 1} a(n)\int_0^\infty e^{-nt}t^{s-1}dt = \sum_{n\geq 1} \frac{a(n)}{n^s}\int_0^\infty e^{-T}T^{s-1}dT = \Gamma(s)L(\chi, s), \tag{6.3}$$

where for the second equality we have made the substitution $T = nt$. We can switch the order of integration and summation in the first equality because we have absolute convergence, which follows from the following linear bound on the $a(n)$'s:

Lemma 6.2. *For all n, $a(n) \leq n$.*

Proof. It is easily seen by induction that for all $m \in \mathbb{N}$, $m + 1 \leq 2^m$, and hence $m + 1 \leq p^m$ for all primes p.

Factor n as $p_1^{m_1}p_2^{m_2}\cdots p_k^{m_k}$. Then, by the results of Section 5, we see $|a(n)| \leq |a(p_1^{m_1})a(p_2^{m_2})\cdots a(p_k^{m_k})| \leq |(m_1 + 1)(m_2 + 1)\cdots(m_k + 1)| \leq |p_1^{m_1}p_2^{m_2}\cdots p_k^{m_k}| = n$. □

For any $N > 0$, (6.3) combined with the asymptotic expansion (6.2) implies that for some $\varepsilon > 0$,

$$\Gamma(s)L(\chi, s) = \int_0^\infty \left(\sum_{n\geq 1} a(n)e^{-nt}\right) t^{s-1}dt = \int_0^\varepsilon \left(\sum_{n\geq 0} c(n)t^n\right) t^{s-1}dt + \int_\varepsilon^\infty \left(\sum_{n\geq 1} a(n)e^{-nt}\right) t^{s-1}dt. \tag{6.4}$$

We truncate our asymptotic expansion to break up the first part of the integral as

$$\int_0^\varepsilon \left(\sum_{n\geq 0} c(n)t^n\right) t^{s-1}dt = \int_0^\varepsilon \sum_{n=0}^{N} c(n)t^{n+s-1}dt + \int_0^\varepsilon O(t^{n+s-1})dt = \sum_{n=0}^{N} c(n)\frac{\varepsilon^{n+s}}{n+s} + F(s).$$

Because $f = O(t^{N+s-1})$ means that for some M, $f \leq Mt^{M+s-1}$, we then have that

$$|F(s)| \leq |M| \left|\int_0^\varepsilon t^{N+s-1}dt\right| = |M| \left|\frac{t^{N+s}}{N+s}\right|\Bigg|_{t=0}^{t=\varepsilon},$$

which is finite for $\Re(s) > -N$. So $F(s)$ is analytic for $\Re(s) > -N$.

Now consider the second half of (6.4), $G(s) = \int_\varepsilon^\infty \left(\sum_{n\geq 1} a(n)e^{-nt}\right) t^{s-1}dt$. By Lemma 6.2, again, the integrand is bounded for any s, and so $G(s)$ is analytic.

So (6.4) becomes

$$\Gamma(s)L(\chi, s) = \sum_{n=0}^{N} c(n)\frac{\varepsilon^{n+s}}{n+s} + F(s) + G(s),$$

where $F(s) + G(s)$ is analytic. Taking residues of both sides, we find

$$c(n) = \frac{(-1)^n}{n!}L(\chi, -n).$$

□

Acknowledgements

We would like to thank J. Anderson and J. Lovejoy for their help and guidance. We would also like to thank R. Chatterjee, G. Coogan, W. McGraw, K. Ono, and J. Rouse for their lectures and assistance. We especially thank K. Ono for organizing the 2003 Number Theory Research Experience for Undergraduates (REU) at the University of Wisconsin at Madison during which this research was conducted.

References

[1] G. Andrews, *q-series: their development and application in analysis, number theory, combinatorics, physics, and computer algebra*. CBMS Regional Conference Series in Mathematics. Amer. Math. Soc. **66** (1986)
[2] T. Apostol, *Introduction to Analytic Number Theory*. Springer, New York. (1984)
[3] G. Andrews, F. Dyson and D. Hickerson, *Partitions and indefinite quadratic forms*. Invent. math. **91**, 391-407 (1988)
[4] G. Andrews and D. Hickerson, *Sixth order mock theta functions*. Adv. Math. **89**, 60-105 (1991)
[5] G. Andrews, J. Jimenez-Urroz, and K. Ono, *q-series identities and values of certain L-functions*. Duke Math. J. **108**, 395-419 (2001)
[6] F. Dyson, *A Walk Through Ramanujan's Garden*. Lecture given at the Ramanujan Centenary Conference, University of Illinois, June 2, 1987. Published in "Ramanujan Revisted," Ed. G. Andrews et al., [San Diego, Academic Press], pp. 7-28 (1988)
[7] J. Lovejoy and K. Ono, *Hypergeometric generating functions for values of Dirichlet and other L-functions*. PNAS. **100**, 6904-6909 (2003)
[8] W. Narkiewicz, *Elementary and Analytic Theory of Algebraic Numbers*. Springer, New York. (1990)
[9] R. Odoni, *On norms of integers in a full module of an algebraic number field and the distribution of values of binary integral quadratic forms*. Mathematika **22** 108-111 (1975)
[10] L. Slater, *Further identities of the Rogers-Ramanujan type*. Proc. London Math. Soc. (2) **54** 147-167 (1950)

111 Bay State Rd
Boston, MA 02215-1700
danl@mit.edu

2324 Lake Pl
Minneapolis, MN 55405-2472
david_favero@brown.edu

1820 Sims Rd NW
Oak Grove, MN 55011-9263
kate@uchicago.edu

CPU 275093 University of Rochester
Rochester, NY 14627-5093
sz005j@mail.rochester.edu