Math 114 Sample Midterm II Key

Disclaimer: These haven't been cross-checked for errors – check your work! Only the relevant problems (as selected by Dr. Stovall) have been addressed in this key.

Part I - True/False

1. *False.* We need the limit to be the same along *every* path through $(0,0)$, not just along straight lines. As an example, define

$$f(x,y) = \begin{cases} \frac{x^4}{x^4+y^2} & y \neq 0, \\ 0 & y = 0 \end{cases}$$

and consider its limit at $(0,0)$. Along the line $y = mx$ for $m \neq 0$ and $x \neq 0$ (so $(x,y) \neq (0,0)$), we have

$$f(x,y) = \frac{x^4}{x^4 + (mx)^2} = \frac{x^4}{x^4 + m^2x^2} = \frac{x^2}{x^2 + m^2}$$

which approaches 0 as $x \to 0$. Therefore f approaches zero as we go to $(0,0)$ along all nonvertical and nonhorizontal lines. To handle the vertical line, described by $x = 0$, note that along that line

$$f(x,y) = f(0,y) = \frac{0}{0+y} = 0 \text{ for } y \neq 0;$$

so f approaches zero along the vertical line as we approach the origin. Finally, look at the horizontal line, described by $y = 0$. We know $f(x,y) = 0$ whenever $y = 0$, so obviously the limit of f as we approach $(0,0)$ along a horizontal line is 0. Altogether then, we conclude that no matter which line we approach the origin along, we see $f(x,y)$ approaching 0.

However, the limit $\lim_{(x,y)\to(0,0)} f(x,y)$ still might be nonzero, or might not exist! Look at the path $y = x^2$. Along this path

$$f(x,y) = \frac{x^4}{x^4 + y^2} = \frac{x^4}{x^4 + x^4} = 1/2 \text{ for } x \neq 0.$$

Therefore as we approach $(0,0)$ along the parabola $y = x^2$, f approaches $1/2$, not zero – so $f(x,y)$ has no limit as $(x,y) \to (0,0)$.

[Yes, I'd like to find a quicker example...]

6. *False.* The given limit just says that $f(x,y)$ is continuous at (a,b); it need not be differentiable. Example: $f(x,y)=\sqrt{x^2+y^2}$, $(a,b)=(0,0)$.

7. *False.* A vector function of constant length always lands on a sphere, or a circle if we're working in two dimensions. This is what motivates choosing the following example:

Let $\mathbf{u}(t)=(\cos t)\mathbf{i}+(\sin t)\mathbf{j}$. This has constant length 1. Differentiating this gives $\mathbf{u}'(t)=(-\sin t)\mathbf{i}+(\cos t)\mathbf{j}$. Now the dot product says

$$\mathbf{u}(t)\cdot\mathbf{u}'(t)=(\cos t)(-\sin t)+(\sin t)(\cos t)=0.$$

Therefore $\mathbf{u}(t)$ and $\mathbf{u}'(t)$ are actually perpendicular! Since neither of these is ever the zero vector, this means they cannot be parallel (the zero vector is parallel to *every* vector).

8. *True.* (See page 930 of textbook – Euler's Theorem.)

Part II - Multiple Choice

1. *e.* We calculate

$$f_{xy} = (f_x)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(\sin y) = \cos y.$$

Evaluating this at $(x, y) = (2, \pi/3)$ gives $\cos(\pi/3) = 1/2$.

2. *d.* The only restriction on the domain here arises because we cannot take the square root of negative numbers. Thus the expression under the square root, $x^2 + y^2 - 1$, must be at least 0. This gives the inequality $x^2 + y^2 \geq 1$, which corresponds to d.

4. *d.* Both numerator and denominator approach 0 as $(x, y) \to (0, 0)$ so we cannot substitute. However, we can use polar coordinates. Use $x = r\cos\theta$, $y = r\sin\theta$; as $(x, y) \to (0, 0)$, r goes to 0, though we cannot say what θ is doing. Thus substituting makes our limit look like

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{\sqrt{x^2 + y^2}} = \lim_{r\to 0^+} \frac{r^2\cos\theta \cdot r\sin\theta}{r} = \lim_{r\to 0^+} (r^2 \cos\theta \sin\theta).$$

We do not know how θ is varying as we near the origin $(0, 0)$ (for instance, we could be spiraling in) BUT we do know that $\cos\theta\sin\theta$ will never get bigger than 1 in absolute value, since $|\sin\theta| \leq 1$ and $|\cos\theta| \leq 1$ always. Therefore no matter what θ does, $r^2\cos\theta\sin\theta$ will approach zero as $r \to 0$; there is no way for the $\cos\theta\sin\theta$ to grow and overpower the r^2 factor that is going to zero.

(Aside #1: if you want to see this formally, take absolute values, using that $|\cos\theta\sin\theta| \leq 1$:

$$\lim_{r\to 0^+} |r^2\cos\theta\sin\theta| \leq \lim_{r\to 0^+} |r^2| = \lim_{r\to 0^+} r^2 = 0.$$

Since absolute values are always at least 0, this limit is also at least 0. We just showed that it is at most 0, so the limit is forced to actually be 0. Finally, if the absolute value of $r^2\cos\theta\sin\theta$ approaches 0 then $r^2\cos\theta\sin\theta$ must itself approach 0; no other number has absolute value 0.)

Another way to see this is that, intuitively, the numerator is a polynomial of degree 3 and the denominator "acts like" a polynomial of degree 1 (since it is the square root of a polynomial of degree 2). Near $(0, 0)$ a higher degree polynomial will go to 0 faster than one of lower degree – for the 1-variable case, look at, e.g., the graphs of $y = x$, $y = x^2$, $y = x^3$. What this means for us here is that the numerator is going to zero faster than the denominator. So, when you divide, the numerator will "overpower" the denominator and the quotient will go to 0. (Aside #2: If it was the other way around, the denominator would be approaching 0 faster than the numerator, so the whole fraction would be growing rather than approaching 0.)

5. *e.* To find the length of the curve, use the length formula

$$L = \int_a^b |\mathbf{v}(t)|\,dt,$$

where $\mathbf{v}(t) = \mathbf{r}'(t)$ and $a \le t \le b$ is the range of t. Here,

$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle 2\cos 2t, -2\sin 2t, 3t^{1/2}\rangle$$

so that

$$\begin{aligned} |\mathbf{v}(t)| &= \sqrt{(2\cos 2t)^2 + (-2\sin 2t)^2 + (3t^{1/2})^2} \\ &= \sqrt{4\cos^2 2t + 4\sin^2 2t + 9t} \\ &= \sqrt{4+9t}. \end{aligned}$$

Our range of t is the interval $[0,1]$, so our length is (using the substitution $u = 4+9t$, $du = 9\,dt$)

$$\begin{aligned} L &= \int_0^1 \sqrt{4+9t}\,dt \\ &= \int_4^{13} \frac{1}{9}\sqrt{u}\,du \\ &= \frac{1}{9}\left(\frac{2}{3}u^{3/2}\Big|_4^{13}\right) \\ &= \frac{1}{9}\cdot\frac{2}{3}(13^{3/2} - 4^{3/2}) \\ &= \frac{2}{27}(13\sqrt{13} - 4\sqrt{4}) \\ &= \frac{2}{27}(13\sqrt{13} - 8). \end{aligned}$$

This is answer e.

6. *e.* The equation of the plane tangent to the graph of $z = f(x, y)$ at (x_0, y_0) is

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0$$

where $z_0 = f(x_0, y_0)$. Here $f(x, y) = e^{2x+2y}$, $(x_0, y_0) = (0, 0)$ and $z_0 = 1$. Partial differentiation gives us

$$f_x = \frac{\partial}{\partial x}(e^{2x+2y}) = \left(\frac{\partial}{\partial x}(2x + 2y)\right)e^{2x+2y} = 2e^{2x+2y}$$

and

$$f_y = \frac{\partial}{\partial y}(e^{2x+2y}) = \left(\frac{\partial}{\partial y}(2x + 2y)\right)e^{2x+2y} = 2e^{2x+2y}.$$

Substituting everything into the equation above gives

$$2e^0(x - 0) + 2e^0(y - 0) - (z - 1) = 0$$

or

$$2x + 2y - z + 1 = 0.$$

This corresponds to e.

7. *b.* To find $\mathbf{r}(t)$, first integrate, but keep in mind constants of integration will pop up:

$$\mathbf{r}(t) = \int \mathbf{r}'(t)\,dt = \left(\int t^2\,dt\right)\mathbf{i} + \left(\int t^3\,dt\right)\mathbf{j} = \frac{t^3}{3}\mathbf{i} + \frac{t^4}{4}\mathbf{j} + \mathbf{C}.$$

(Here $\mathbf{C}$ is a constant vector that comes from the constants of integration for each integral.)

Now substitute $t = 0$ and use that $\mathbf{r}(0) = \mathbf{i}$ (given):

$$\mathbf{i} = \mathbf{r}(0) = 0\mathbf{i} + 0\mathbf{j} + \mathbf{C} = \mathbf{C}.$$

Since $\mathbf{C} = \mathbf{i}$,

$$\mathbf{r}(t) = \frac{t^3}{3}\mathbf{i} + \frac{t^4}{4}\mathbf{j} + \mathbf{i} = \left(\frac{t^3}{3} + 1\right)\mathbf{i} + \frac{t^4}{4}\mathbf{j}.$$

Therefore

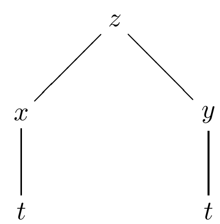
$$\mathbf{r}(1) = \left(\frac{1}{3} + 1\right)\mathbf{i} + \frac{1}{4}\mathbf{j} = \frac{4}{3}\mathbf{i} + \frac{1}{4}\mathbf{j}.$$

8. *e.* This is an exercise in careful differentiation. We'll do two methods.

(a) (Chain rule for multiple variables.) We are given that x and y are both functions of independent variable t. Also, the equation for z gives z as a function of x and y. Therefore by the chain rule, we have

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}.$$

(You may see this represented in the textbook via this [oversized...] diagram:)



Now we just have to evaluate each part of the equation above at $t = 1$. First we do the partial differentiation and get

$$\frac{\partial z}{\partial x} = y^2 + 3x^2 y, \qquad \frac{\partial z}{\partial y} = 2xy + x^3.$$

Since at $t = 1$ we have $x(1) = 1$ and $y(1) = 2$, we can just substitute to get

$$\left.\frac{\partial z}{\partial x}\right|_{t=1} = 2^2 + 3 \cdot 1^2 \cdot 2 = 10, \qquad \left.\frac{\partial z}{\partial y}\right|_{t=1} = 2 \cdot 1 \cdot 2 + 1^3 = 5.$$

We also know that $x'(1) = 3$ and $y'(1) = 4$, so

$$\left.\frac{dx}{dt}\right|_{t=1} = 3, \qquad \left.\frac{dy}{dt}\right|_{t=1} = 4.$$

Substituting these in the first equation we get

$$\left.\frac{dz}{dt}\right|_{t=1} = 10 \cdot 3 + 5 \cdot 4 = 50.$$

(b) (Direct differentiation.) We can differentiate the equation for z in terms of t directly, BUT (!!!) we have to keep in mind that x and y are functions of t.

$$\begin{aligned}
\frac{dz}{dt} &= \frac{d}{dt}(xy^2 + x^3 y) \\
&= \frac{d}{dt}(xy^2) + \frac{d}{dt}(x^3 y) \\
&= \frac{dx}{dt} \cdot y^2 + x \cdot \frac{d}{dt}(y^2) + \left(\frac{d}{dt}(x^3)\right) \cdot y + x^3 \cdot \frac{dy}{dt}.
\end{aligned}$$

Before continuing, note that in order to differentiate x^3, for instance, we have to use the chain rule because x is a function of t; thus $\frac{d}{dt}(x^3) = 3x^2x'$, where x' means $x'(t)$. So, continuing the differentiation with this in mind...

$$\begin{aligned}\frac{dz}{dt} &= \frac{dx}{dt} \cdot y^2 + x \cdot \frac{d}{dt}(y^2) + \left(\frac{d}{dt}(x^3)\right) \cdot y + x^3 \cdot \frac{dy}{dt} \\ &= x'y^2 + x \cdot 2y \cdot y' + 3x^2 \cdot x' \cdot y + x^3y'.\end{aligned}$$

At $t = 1$, this is equal to (by substituting the given values of x, x', y, y')

$$3 \cdot 2^2 + 1 \cdot 2 \cdot 2 \cdot 4 + 3 \cdot 1^2 \cdot 3 \cdot 2 + 1^3 \cdot 4 = 12 + 16 + 18 + 4 = 50.$$

9. *b.* The normal component of the acceleration is given by

$$a_N = \kappa|\mathbf{v}|^2$$

where κ is curvature of the curve and $|\mathbf{v}|$ is the speed of the particle. To compute both of these, we differentiate twice:

$$\begin{aligned}\mathbf{r}(t) &= t\mathbf{i} + t^2\mathbf{j}, \\ \mathbf{v}(t) = \mathbf{r}'(t) &= \mathbf{i} + (2t)\mathbf{j}, \\ \mathbf{a}(t) = \mathbf{a}'(t) &= 2\mathbf{j}.\end{aligned}$$

Now the speed in terms of t is

$$|\mathbf{v}(t)| = \sqrt{1^2 + (2t)^2} = \sqrt{1 + 4t^2}.$$

To compute curvature, we use the formula

$$\kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3}.$$

To compute cross product we have to plug in zero for the third coordinate:

$$\mathbf{v}(t) \times \mathbf{a}(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2t & 0 \\ 0 & 2 & 0 \end{vmatrix} = \mathbf{k} \begin{vmatrix} 1 & 2t \\ 0 & 2 \end{vmatrix} = 2\mathbf{k}$$

so $|\mathbf{v} \times \mathbf{a}| = 2$ $(= \sqrt{0^2 + 0^2 + 2^2}.)$

Now speed at $t = 1$ is $\sqrt{1 + 4 \cdot 1^2} = \sqrt{5}$ and curvature at $t = 1$ is

$$\kappa = \frac{2}{(\sqrt{1 + 4t^2})^3} = \frac{2}{(\sqrt{5})^3} = \frac{2}{5\sqrt{5}},$$

so

$$a_N = \kappa|\mathbf{v}|^2 = \frac{2}{5\sqrt{5}} \cdot (\sqrt{5})^2 = \frac{2}{5\sqrt{5}} \cdot 5 = \frac{2}{\sqrt{5}}.$$

11. *e.* We can do this by implicit differentiation; keep in mind throughout that y is a function of x.

$$\begin{aligned}
& & x^3 + y^3 &= x^2y^2 \\
&\Rightarrow & \frac{d}{dx}(x^3 + y^3) &= \frac{d}{dx}(x^2y^2) \\
&\Rightarrow & 3x^2 + \frac{d}{dx}(y^3) &= 2xy^2 + x^2\frac{d}{dx}(y^2) \\
&\Rightarrow & 3x^2 + 3y^2\frac{dy}{dx} &= 2xy^2 + x^2 \cdot 2y\frac{dy}{dx} \\
&\Rightarrow & 3y^2\frac{dy}{dx} - 2x^2y\frac{dy}{dx} &= 2xy^2 - 3x^2 \\
&\Rightarrow & \frac{dy}{dx}(3y^2 - 2x^2y) &= 2xy^2 - 3x^2 \\
&\Rightarrow & \frac{dy}{dx} &= \frac{2xy^2 - 3x^2}{3y^2 - 2x^2y}.
\end{aligned}$$

At $(x, y) = (2, 2)$ this equals

$$\frac{2 \cdot 2 \cdot 2^2 - 3 \cdot 2^2}{3 \cdot 2^2 - 2 \cdot 2^2 \cdot 2} = \frac{16 - 12}{12 - 16} = -1.$$