Math 114 Sample Midterm II Key (Part 2)

Part I: True/False

2. *True.* (If f has a local maximum at (a,b) then it has a critical point there; i.e. $f_x(a,b) = f_y(a,b) = 0$. But then the gradient is $\nabla f(a,b) = \langle f_x(a,b), f_y(a,b)\rangle = \langle 0,0\rangle$.)

3. *True.* (The given point is a critical point and $D = f_{xx}f_{yy} - f_{xy}^2 < 0$ there according to the given inequality. Therefore f has a saddle point at $(2,1)$ by the second derivative test.)

4. *True.* (If $f(x,y) = \sin x + \sin y$ then $\nabla f(x,y) = \langle \cos x, \cos y\rangle$. Each coordinate of ∇f can only be at most 1 in absolute value. Therefore the length of ∇f is

$$|\nabla f(x,y)| = \sqrt{(\cos x)^2 + (\cos y)^2} \leq \sqrt{1+1} = \sqrt{2}$$

always. Now for any unit vector $\mathbf{u}$, the directional derivative of f in the direction of $\mathbf{u}$ is

$$D_{\mathbf{u}}f(x,y) = \nabla f(x,y)\cdot \mathbf{u} = |\nabla f(x,y)||\mathbf{u}|\cos\theta = |\nabla f|\cdot 1\cdot \cos\theta = |\nabla f|\cos\theta$$

where θ is the angle between the vectors $\nabla f(x,y)$ and $\mathbf{u}$. Now $\cos\theta$ is between -1 and 1 and $0 \leq |\nabla f| \leq \sqrt{2}$, so $-\sqrt{2} \leq D_{\mathbf{u}}f(x,y) \leq \sqrt{2}$.)

5. *False.* The intuition here is to first find a function $g(x)$ (of one variable) that has at least 2 local minima; it may also have some local maxima. Next we use a function $h(y)$ (again of one variable) that has just one local min and no local maxima. (Here we use $g(x) = \cos x$ and $h(y) = y^2$.) If we set $f(x,y) = g(x) + h(y)$, then the cross sections in one $x-y$ axis direction will look like graphs of $g(x)$ and the cross sections in the other direction will look like graphs of $h(y)$. If you picture this (could be difficult!) then when $g(x)$ and $h(y)$ both have minima (low points in the graph), so will $f(x,y)$. That's how we'll get (at least) 2 local minima. If $f(x,y)$ has a local maximum (a high point in the graph) then if we take cross sections in both x and y directions, we should get high points on each of those graphs. But the graph of $h(y)$ will never have any high points! So nowhere will we find a local maximum for $f(x,y)$. [You can work this all out formally by using the second derivative test for max/min with single variable functions.]

On to the work: set $f(x,y) = \cos x + y^2$. Take first and second order partial derivatives:

$$f_x = -\sin x,\ f_y = 2y,\ f_{xx} = -\cos x,\ f_{yy} = 2,\ f_{xy} = 0.$$

Critical points occur when f_x and f_y are both zero; f_x is zero exactly when x is a multiple of π (so that $\sin x = 0$), f_y is zero exactly when $y = 0$. For the second derivative test,

$$D = f_{xx}f_{yy} - f_{xy}^2 = -2\cos x.$$

At any critical point, $\sin x = 0$ so since $\sin^2 x + \cos^2 x = 1$, that forces $\cos x$ to be ± 1.

If $\cos x = 1$, then $D = -2 < 0$ and we have a saddle point. If $\cos x = -1$, then $D = 2 > 0$ and $f_{xx} = -\cos x = -(-1) = 1$ so we have a local minimum. Therefore all critical points are saddle points or local minima; there are no local maxima. Finally we just need to show that we actually have two local minima. These occur when $\cos x = -1$ (and $f_y = 2y = 0$ or $y = 0$). So, for example, $(x, y) = (-\pi, 0)$ and $(x, y) = (\pi, 0)$ are both local minima. Now $f(x, y)$ has (at least) 2 local minima but no local maxima.

9. *True.*

10. *False.* The given limits of integration correspond to the ranges $0 \leq x \leq 1$, $0 \leq y \leq 1 - x^2$, which does define the correct region. However, the order of integration is incorrect! Notice that if you integrate the inside integral first, the dx disappears and you are still left with $1 - x^2$ floating around on the outside integral... if you complete the integration you won't even get a number. General rule of thumb: the limits of integration of the outside integral should NEVER be in terms of the variables of integration, x and y in this case (though you can have other external variables there).

Part II: Multiple Choice

3. *Answer is 108, not among the choices.* The integral we are computing is

$$\iint_R f(x,y)\,dA = \iint_R (x^2 + 2xy)\,dA.$$

Since we are in rectangular coordinates, $dA = dx\,dy$ (in any order). The region of integration is just the rectangle $0 \le x \le 3$, $0 \le y \le 4$ so the integral is

$$\begin{aligned}
\int_0^3 \int_0^4 (x^2 + 2xy)\,dy\,dx &= \int_0^3 \left[x^2 y + xy^2\right]_{y=0}^{y=4} dx \\
&= \int_0^3 (4x^2 + 16x)\,dx \\
&= \left[\frac{4x^3}{3} + 8x^2\right]_0^3 \\
&= \frac{4 \cdot 3^3}{3} + 8 \cdot 3^2 \\
&= 4 \cdot 9 + 8 \cdot 9 = \mathbf{108}.
\end{aligned}$$

If we integrate in the other order we get

$$\begin{aligned}
\int_0^4 \int_0^3 (x^2 + 2xy)\,dx\,dy &= \int_0^4 \left[\frac{x^3}{3} + x^2 y\right]_{x=0}^{x=3} dy \\
&= \int_0^4 \left(\frac{27}{3} + 9y\right) dy \\
&= 9 \int_0^4 (1 + y)\,dy \\
&= 9\left[y + \frac{y^2}{2}\right]_0^4 \\
&= 9(4 + 4^2/2) = 9 \cdot 12 = \mathbf{108}.
\end{aligned}$$

10. *a.* We are finding the directional derivative of $f(x, y, z) = xe^{xy/z}$ in the direction of $\mathbf{v} = -3\mathbf{i} + 6\mathbf{j} + 6\mathbf{k}$ at the point $P = (3, 0, 1)$. We know that this directional derivative is given by

$$D_{\mathbf{u}}f(P) = D_{\mathbf{u}}f(3, 0, 1) = \nabla f|_P \cdot \mathbf{u}$$

where $\mathbf{u}$ is the *unit* vector in the direction of $\mathbf{v}$. Let's compute each component of this.

First, $\mathbf{u} = \mathbf{v}/|\mathbf{v}|$. We compute

$$|\mathbf{v}| = \sqrt{(-3)^2 + 6^2 + 6^2} = \sqrt{9 + 36 + 36} = \sqrt{81} = 9.$$

Therefore $\mathbf{u} = \mathbf{v}/9 = (-1/3)\mathbf{i} + (2/3)\mathbf{j} + (2/3)\mathbf{k}$.

Next we need ∇f, which means we need the partial derivatives of f. By the product rule,

$$f_x = \frac{\partial}{\partial x}(x) \cdot e^{xy/z} + x \cdot \frac{\partial}{\partial x}(e^{xy/z}).$$

The first summand is easily handled (the derivative is 1). For the second we use the chain rule to get

$$\frac{\partial}{\partial x}(e^{xy/z}) = \frac{y}{z}e^{xy/z}$$

so that

$$f_x = e^{xy/z} + \frac{xy}{z}e^{xy/z}.$$

Next, by chain rule,

$$f_y = x \cdot \frac{\partial}{\partial y}(e^{xy/z}) = x \cdot \frac{x}{z}e^{xy/z} = \frac{x^2}{z}e^{xy/z}.$$

Again by the chain rule,

$$f_z = x \cdot \frac{\partial}{\partial z}(e^{xy/z}) = xe^{xy/z}\frac{\partial}{\partial z}\left(\frac{xy}{z}\right) = xe^{xy/z} \cdot xy \cdot -z^{-2}.$$

We compute

$$\begin{aligned}
f_x(3, 0, 1) &= e^0 + 0 \cdot e^0 = 1, \\
f_y(3, 0, 1) &= \frac{3^2}{1}e^0 = 9, \\
f_z(3, 0, 1) &= 0 \text{ (since } y = 0).
\end{aligned}$$

Therefore $\nabla f|_P = \nabla f(3, 0, 1) = \langle 1, 9, 0 \rangle$.

Finally, the directional derivative is

$$D_{\mathbf{u}}f(3, 0, 1) = \nabla f|_P \cdot \mathbf{u} = \langle 1, 9, 0 \rangle \cdot \langle -\frac{1}{3}, \frac{2}{3}, \frac{2}{3} \rangle = -\frac{1}{3} + \frac{18}{3} = \mathbf{\frac{17}{3}}.$$

12. *f.* The ranges on the variables of integration x from 0 to y, y from 0 to 1, z from 0 to 1. Since the range for x depends on y we have to integrate x before we integrate y; i.e., dx should occur before dy in the triple integral (remember, when we evaluate multiple integrals we integrate the innermost integral first). Since we are in rectangular coordinates, $dV = dx\,dy\,dz$ (in any order). We will integrate in the order $dx\,dy\,dz$. Compute:

$$\begin{aligned}\int_0^1\int_0^1\int_0^y x^2\,dx\,dy\,dz &= \int_0^1\int_0^1\left[\frac{x^3}{3}\right]_{x=0}^{x=y} dy\,dz\\ &= \int_0^1\int_0^1 \frac{y^3}{3}\,dy\,dz\\ &= \int_0^1\left[\frac{y^4}{12}\right]_{y=0}^{y=1} dz\\ &= \int_0^1 \frac{1}{12}\,dz = \mathbf{\frac{1}{12}}.\end{aligned}$$

If we want we can also integrate in the order $dz\,dx\,dy$:

$$\begin{aligned}\int_0^1\int_0^y\int_0^1 x^2\,dz\,dx\,dy &= \int_0^1\int_0^y \left[x^2 z\right]_{z=0}^{z=1} dx\,dy\\ &= \int_0^1\int_0^y x^2\,dx\,dy\\ &= \int_0^1\left[\frac{x^3}{3}\right]_{x=0}^{x=y} dy\\ &= \int_0^1 \frac{y^3}{3}\,dy\\ &= \left[\frac{y^4}{12}\right]_0^1 = \mathbf{\frac{1}{12}}.\end{aligned}$$

Part III: Free Response

1. Here we seek to maximize the volume $V = xyz$ of a rectangular box subject to the restriction $2x + 2y + z = 3$ where x, y, z are the dimensions of the box in meters. We will do this via Lagrange multipliers.

The function we are maximizing is $f(x, y, z) = xyz$ subject to the constraint $g(x, y, z) = 2x + 2y + z - 3 = 0$. We compute gradients.

$$f_x = yz, \ f_y = xz, \ f_z = xy$$

so

$$\nabla f(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k} = \langle yz, xz, xy \rangle;$$

also, $g_x = 2$, $g_y = 2$, $g_z = 1$ so that

$$\nabla g(x, y, z) = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k} = \langle 2, 2, 1 \rangle.$$

Now maximize f with the constraint $g = 0$ we must solve $\nabla f = \lambda \nabla g$. Matching up coordinates gives three simultaneous equations

$$yz = 2\lambda, \ xz = 2\lambda, \ xy = \lambda.$$

If we substitute the last into the first, we get $yz = 2xy$. You might be tempted to say $z = 2x$ right away, but keep in mind we can only divide both sides by y if $y \neq 0$, so we really get $y = 0$ OR $z = 2x$. Of course here we're trying to maximize a volume and y is the length of one side of a box, so we can automatically discard the possibility $y = 0$ and say $z = 2x$. If we substitute the last of the three equations into the second one we get $xz = 2xy$ so $x = 0$ or $z = 2y$. Again, x can't be zero, so $z = 2y$.

Now $2z + 2y + z = 3$ (this is the constraint) and $z = 2y = 2x$, so substitution gives $z + z + z = 3z = 3$ and $z = 1$. Since $z = 1$ we get $x = 1/2$, $y = 1/2$. Therefore $(x, y, z) = (1/2, 1/2, 1)$ maximizes the volume of the box, and the resulting maximum volume is $(1/2)(1/2) \cdot 1 = 1/4$ m^3. The value of λ that solves the above system of equations is $\lambda = 1/4$. (The significance of the λ is that it, along with the x, y, z, must satisfy the simultaneous equations that come from $\nabla f = \lambda \nabla g$.)

2. a) The limits of integration in the double integral tell us that the region of integration is given by $0 \le x \le 1$ and $0 \le y \le \sqrt{1-x^2}$. The graph of $y = \sqrt{1-x^2}$ is the top half of a circle of radius 1 centered at the origin. Therefore the region of integration is the part of the disk of radius 1 (in other words, the circle of radius 1 together with all the points inside it) centered at the origin that is in the first quadrant.

2. b) We are to evaluate

$$\int_0^1 \int_0^{\sqrt{1-x^2}} e^{-(x^2+y^2)}\, dy\, dx.$$

As-is, we cannot evaluate this integral. The region of integration is a quarter-disk and $x^2 + y^2$ shows up in the integral, so it is reasonable to try to convert this into polar coordinates.

To find the ranges of r and θ for a new double integral, we just have to describe the region of integration in terms of r and θ. Since we're taking all first quadrant points inside and on the circle of radius 1 (centered at $(0,0)$) we get $0 \le r \le 1$ and $0 \le \theta \le \pi/2$, the latter because we are in the first quadrant.

As to the integrand, we know $x^2 + y^2 = r^2$, so $e^{-(x^2+y^2)} = e^{-r^2}$. Also, $dy\, dx = dA = r\, dr\, d\theta = r\, d\theta\, dr$. (Remembering the extra factor of r is key.)

Therefore the integral becomes (if we integrate in the order $dr\, d\theta$)

$$\int_0^{\pi/2} \int_0^1 e^{-r^2} r\, dr\, d\theta.$$

Evaluate the inner integral with the substitution $u = r^2$, $du = 2r\, dr$:

$$\begin{aligned}
\int_0^1 e^{-r^2} r\, dr &= \int_{0^2}^{1^2} e^{-u} \cdot \frac{1}{2}\, du \\
&= \frac{1}{2} \int_0^1 e^{-u}\, du \\
&= \frac{1}{2} \Big[-e^{-u} \Big]_0^1 \\
&= \frac{1}{2}(-e^{-1} + e^0) \\
&= \frac{1}{2}\left(1 - \frac{1}{e}\right).
\end{aligned}$$

Plugging this into the polar double integral then gives

$$\int_0^{\pi/2} \int_0^1 e^{-r^2} r\, dr\, d\theta = \int_0^{\pi/2} \frac{1}{2}\left(1 - \frac{1}{e}\right) d\theta = \frac{1}{2}\left(1 - \frac{1}{e}\right) \int_0^{\pi/2} d\theta = \frac{\pi}{4}\left(\mathbf{1} - \frac{\mathbf{1}}{\mathbf{e}}\right).$$

3. If R is a region in the x-y plane and $f(x, y)$ is a function that is greater than 0 for (x, y) inside R (we need the graph of $f(x, y)$ to lie above the x-y plane), then the volume of the solid between the graph of $z = f(x, y)$ above R and the region R itself is

$$V = \iint_R f(x, y)\, dA = \iint_R f(x, y)\, dx\, dy = \iint_R f(x, y)\, dy\, dx.$$

We use this formula to construct a couple integrals.

For both a) and b) we are integrating over a region in the x-y plane. Since we're staying in the first octant here, $x \geq 0$ and $y \geq 0$. The top of our solid is the graph of the function $z = f(x, y) = 1 - y - x^2$. The part of the x-y plane below our solid is exactly the part where $f(x, y) \geq 0$; this is where the graph lies above the x-y plane. Therefore our region R is given by three inequalities:

$$x \geq 0,\ y \geq 0,\ f(x, y) = 1 - y - x^2 \geq 0 \ \ (\text{or } y \leq 1 - x^2).$$

This region is the region under the parabola $y = 1 - x^2$ in the first quadrant of the x-y plane. (You may want to sketch this to understand what's coming up.)

We can describe R in 2 different ways, either by reading off the range of x first or the range of y first.

From the graph of R we have $0 \leq x \leq 1$. Now getting y in terms of x is easy: R is between the x-axis and the graph of $y = 1 - x^2$ so $0 \leq y \leq 1 - x^2$. We can put these into an integral... remember, the outer integral's limits cannot contain x or y:

$$V = \int_0^1 \int_0^{1-x^2} f(x, y)\, dy\, dx = \int_0^1 \int_0^{1-x^2} (1 - y - x^2)\, dy\, dx.$$

This takes care of part b).

For part a), we first read off $0 \leq y \leq 1$ from the drawing or description of R. Since $y \leq 1 - x^2$, we have $1 - y \geq x^2$. We're staying in the first quadrant so we only care about nonnegative x; therefore we can take square roots and see $\sqrt{1 - y} \geq x$. (Note that this matches the picture of R, since the graph of the part of the parabola $y = 1 - x^2$ in the first quadrant of the x-y plane can be written as $x = \sqrt{1 - y}$.) So we get the range $0 \leq x \leq \sqrt{1 - y}$ for x. Now put these into an integral to complete part a):

$$V = \int_0^1 \int_0^{\sqrt{1-y}} f(x, y)\, dx\, dy = \int_0^1 \int_0^{\sqrt{1-y}} (1 - y - x^2)\, dx\, dy.$$

4. We are finding the *absolute* maximum and minimum values of $f(x,y) = 3 + xy - x - 2y$ on the closed triangular region R with vertices $(1,0)$, $(5,0)$, $(1,4)$.

First step: Find critical points of f in the interior of R. To do this, we need to find where $f_x = 0$ and $f_y = 0$. Partial differentiation tells us

$$f_x = y - 1, \;\; f_y = x - 2.$$

Both of these are zero exactly at the single point $(2,1)$, so $(2,1)$ is a critical point of f. HOWEVER, it is only relevant if it is in the interior of our region – critical points of f outside R play no role in finding the absolute max and min of f inside R. Here, this point is in R and so will be considered further. (To verify this, find that the equation of the triangle's diagonal is $y = 5 - x$; then the interior of R is described by $1 < x < 5$, $0 < y < 5 - x$. The point $(x,y) = (2,1)$ satisfies both of these inequalities.)

To see what kind of point $(2,1)$ is, we need the second derivative test. Here $f_{xx} = 0$, $f_{yy} = 0$, $f_{xy} = 1$, so $D = f_{xx}f_{yy} - f_{xy}^2 = -1 < 0$. Therefore $(2,1)$ is a saddle point! Since it is not the location of a local min or a local max, we won't find an absolute max or min there! So we can discard the point.

(Alternately, we could just compare the value of f at $(2,1)$ with the values of f at other candidate min/max points we find later and forego the second derivative test.)

Second step: Find maxima and minima of f on boundary of R. First we note the endpoints of each edge, namely $(1,0)$, $(5,0)$, $(1,4)$:

$$\begin{aligned} f(1,0) &= 3 + 1\cdot 0 - 1 - 2\cdot 0 = 2, \\ f(5,0) &= 3 + 5\cdot 0 - 5 - 2\cdot 0 = -2, \\ f(1,4) &= 3 + 1\cdot 4 - 1 - 2\cdot 4 = -2. \end{aligned}$$

We have to find the maxima and minima of f along each edge of R now.

The bottom edge is given by $y = 0$, $1 \le x \le 5$. For $y = 0$

$$f(x,y) = f(x,0) = 3 - x.$$

Differentiating this with respect to x gives -1 which is never equal to zero so by the first derivative test for single variable functions we have no max/min except at endpoints of the interval $0 \le x \le 5$, which we already accounted for.

The left edge is given by $x = 1$, $0 \le y \le 4$. For $x = 0$

$$f(x,y) = 3 - 2y.$$

Differentiating this with respect to y gives $-2 \neq 0$ so there are no max/min of this function along the left edge except at the endpoints.

The diagonal is given by $y = 5 - x$, $1 \le x \le 5$. Along this line

$$\begin{aligned} f(x,y) = f(x, 5-x) &= 3 + x(5-x) - x - 2(5-x) \\ &= 3 + 5x - x^2 - x - 10 + 2x = -x^2 + 6x - 7. \end{aligned}$$

Differentiating this with respect to x gives $-2x+6$, which is zero at $x=3$. Therefore f could have an absolute max or min on the diagonal where $x=3$. Since the diagonal is $y=5-x$, this interesting point is $(3,2)$. We find

$$f(3,2) = 3 + 3 \cdot 2 - 3 - 2 \cdot 2 = 3 + 6 - 3 - 4 = 2.$$

Third step: Compare values of f at all noted points to find absolute max/min. We have $f(1,0) = 2$, $f(5,0) = -2$, $f(1,4) = -2$, and $f(3,2) = 2$. Therefore the maximum value of f on R is 2, achieved at $(3,2)$ and $(1,0)$ and the minimum value of f on R is -2, achieved at $(5,0)$ and $(1,4)$.

(If we did not check the saddle point $(2,1)$ above we could check it now:

$$f(2,1) = 3 + 2 \cdot 1 - 2 - 2 \cdot 1 = 3 + 2 - 2 - 2 = 1.$$

This is not above 2 or below -2 and so doesn't give an absolute max/min.)