Oral Exam Questions - Alina Badus (2006)

Differential Geometry

1. Say S^2 has a Riemannian metric of strictly positive curvature. Prove that any two simple closed geodesics must intersect.

2. Let α, β and γ be the interior angles of a geodesic triangle T in $S^2(1)$. Give an elementary proof that $\alpha+\beta+\gamma-\pi = area(T)$, without using the Gauss-Bonnet theorem.

3. Consider a smooth strictly convex ($K > 0$) closed surface $M^2 \subset \mathbf{R}^3$ and let γ be a simple closed geodesic on M^2. Show that the image of γ under the Gauss map is a "long" curve on S^2.

4. (a) Let M^2 be a minimal surface in $\mathbf{R}^3$. Consider a point $p \in M$ and assume that the Gaussian curvature $K(p) = -4$. What is the distance from p to the closest focal point?

(b) Let $M^3 \subset \mathbf{R}^4$ be a minimal submanifold with constant sectional curvature K. What is K?

5. (a) Define conjugate point, conjugate locus, cut point, cut locus.

Consider an arbitrary Riemannian metric on S^2 and let $p \in S^2$.

(b) Must every geodesic from p have a cut point?

(c) Must every geodesic from p have a conjugate point?

(d) Must some geodesic from p have a conjugate point?

(e) On a closed manifold, can you have an empty conjugate locus, but non-empty cut locus? Examples?

(f) Viceversa, can you have an empty cut locus but non-empty conjugate locus?

(g) Is there a Riemannian metric on $S^2 \times S^2$ with a cut locus homeomorphic to S^2?

6. Let M^n be a complete Riemannian manifold with constant sectional curvature $K = 1$.

(a) Are there bounds on the size of the fundamental group $\pi_1(M)$?

(b) Suppose in odd dimensions that you know the injectivity radius of M to be bigger than some number $i_0 > 0$. Now what is an upper bound on the size of $\pi_1(M)$?

7. Let M^n be a complete Riemannian manifold with Gaussian curvature K; we know $1 \leq K \leq 10$. Report on:

(a) the topology of M;

(b) the injectivity radius of M;

(c) the diameter of M;

(d) the volume of M.

8. Let G be a connected Lie group with a left-invariant metric. Show that the metric is bi-invariant if and only if $\langle [A, B], C\rangle = \langle A, [B, C]\rangle$ for all left-invariant vector fields A, B, C.