Oral Exam Questions - Chenxu He (2006)

Differential Geometry

1. *Killing vector fields*

(a) Let X be a vector field on a Riemannian manifold M. What does it mean to say that X is a Killing field?

(b) Prove that X is a Killing field if and only if $\langle \nabla_Y X, Z\rangle + \langle Y, \nabla_Z X\rangle = 0$ for all vector fields Y and Z on X. Note that only X is being differentiated, not Y or Z.

(c) Use this to prove that a linear vector field $X = a^i_j x^j \partial/\partial x^i$ on $\mathbb{R}^n$ is a Killing field if and only if the matrix $A = (a^i_j)$ of coefficients is skew-symmetric.

(d) Suppose that M is a connected Riemannian manifold and that X is a Killing field on M. Suppose there is a point $p \in M$ at which $X(p) = 0$ and $\nabla_Y X$ vanishes at p for all $Y(p) \in T_pM$. Prove that $X \equiv 0$.

2. *Parallel transport*

(a) Suppose M is a Riemannian manifold with the property that parallel transport of vectors between any two points on M does not depend on the curve used. Prove that the curvature tensor $R(X,Y)Z$ is identically zero.

(b) Suppose we only know that parallel transport of vectors around all closed loops based at a single point p does not depend on the choice of the loop. Prove that the curvature tensor $R(X,Y)Z$ is identically zero.

3. *Submanifolds*

Suppose that M^n is a complete, connected Riemannian manifold and that $N^r \subset M^n$ is a submanifold, with normal bundle νN.

(a) Suppose that N^r is compact. Define the exponential map $\exp : \nu N \to M$ and show that it is onto.

(b) Derive the first variation of arc length formula and show how it is used in your answer to (a).

(c) Suppose M^n is not complete. Is $\exp : \nu N \to M$ still onto?

(d) Suppose again that M^n is complete, but that N^r is only known to be complete, not compact. Is $\exp : \nu N \to M$ still onto?

4. *Surfaces in* $\mathbb{R}^3$

Let M^2 be a smooth surface in $\mathbb{R}^3$.

(a) Suppose we know that through each point $p \in M^2$, there is a straight line in $\mathbb{R}^3$ which lies in M^2. What does this tell you about the curvature of M^2?

(b) Suppose two lines in $\mathbb{R}^3$ go through each point $p \in M^2$ and lie in M^2. Now what do we know about M^2?

(c) Suppose three lines in $\mathbb{R}^3$ go through each point of M^2 and lie in M^2. Now what do we know about M^2?