POWERS:

1. Weak-operator topology:

   (i) Describe the weak operator topology on $B(H)$ in terms of convergence. What is THE key fact about this topology?

   (ii) If $A_n \to A$ weakly, do we have a bound on $\sup_{n\in\mathbb{N}} ||A_n||$? Why or why not?

   (iii) Suppose H has a countable orthonormal basis $\{e_1, e_2, \ldots\}$, and suppose that $\{A_n\}$ is a sequence of elements in $B(H)$ with the property that for every i and j we have

   $$\langle A_n(e_i), e_j\rangle \to \langle A(e_i), e_j\rangle$$

   for some $A \in B(H)$. Does it follow that A_n converges weakly to A? Is $\sup_{n\in\mathbb{N}} ||A_n||$ necessarily bounded? Proof or counterexample.

   (iv) Say there is a linear functional ℓ on $B(H)$ such that for some weak neighborhood of 0 (say $\mathcal{N}(0; f_1, \ldots, f_n, g_1, \ldots, g_n; \epsilon)$) we have

   $$|\ell(A)| < \delta$$

   for all $A \in \mathcal{N}$. What can we say about ℓ?

2. (Shift Operators) Let $H = \ell_2(\mathbb{N})$ and consider the right shift U on H, so $U(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots)$.

   (i) What is the spectrum of U?

   (ii) Give an explicit sequence of unitary elements $\{U_n\} \subset B(H)$ such that U_n converges weakly to U, or prove that no such sequence exists.

   (iii) What is U^*, and what is $I - U^*U$? What can we say about the von Neumann algebra generated by I, U, and U^*?

BLOCK:

3. (Be a man) Is the set J of invertible elements in $B(H)$ dense in $B(H)$? Proof or counterexample.

4. (Spectrum of operators) Say X, Y are bounded operators in some Banach algebra $\mathfrak{B}$. Is it possible to have $XY - YX = I$? Give an example or prove that no such example exists.

5. (Be a REAL man) Say $X \in \mathfrak{B}$ is bounded and is NOT normal, and suppose p is a polynomial. Show that, if $\epsilon > 0$ is sufficiently small and if $||p(X)|| < \epsilon$, then there is some $Y \in \mathfrak{B}$ such that $p(Y) = 0$.

   (Warm-up problem: say X is normal and $||X^2 - X|| < \epsilon$. Show that if certain assumptions about the spectrum of X are made, and if we are allowed to make ϵ sufficiently small, then there exists a non-trivial projection in $\mathfrak{B}$ [i.e. a projection that is neither 0 nor I]. Then find an explicit ϵ that guarantees the above).

6. (Maximal ideals) What is the maximal ideal space of $C(\mathbb{R}^2)$?

GLUCK:

7. Curves in $\mathbb{R}^3$:

   (i) Say α is a smooth curve in $\mathbb{R}^3$ such that, for every point $p \in \alpha$, there is some open neighborhood N_p of p in α such that N_p is contained in a plane. Does α necessarily lie in a plane? Proof or counterexample.

   (ii) Say α is a smooth, closed curve in $\mathbb{R}^3$ with curvature $\kappa > 0$ everywhere. Is α necessarily simple? If so, prove it. If not, give a counterexample and then find a sufficient condition (constant κ with a nice parametrization does NOT count as such a condition) for our smooth closed α to be simple.

   (iii) (Herman forewarns you that this is a topology question, NOT a differential geometry question, so think topologically):

   Say α is a knotted curve in $\mathbb{R}^3$ (i.e., there is no homemorphism $\phi : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ such that $\phi(\alpha) = S^1$). Prove that for every line ℓ in $\mathbb{R}^3$, there exist $p, q \in \alpha$ such that the line joining p to q is parallel to ℓ.

8. Cut loci:

   (i) Define cut locus, and give some standard examples of cut loci on various manifolds.

   (ii) Can non-homeomorphic closed surfaces M and N have homeomorphic cut loci? What if we require that M and N are both orientable or that both are non-orientable? Proof or counterexample in each case.

   (iii) Can non-homeomorphic closed 3-manifolds M and N have homeomorphic cut loci? If not, why? If so, give an example of such manifolds M and N and prove that their cut loci are homeomorphic.