Oral Exam Questions - Ben Schak (2006)

The order of questions was CC1, RP1, HG1, CC2, RP2, RP3, HG2.

Differential Geometry (HG)

1) Let S be a round sphere in 3-space. Does there exist a surface whose normals are all tangent to S?

2a) What is the fundamental group of the space of isometries of a round n-sphere?

Here are some follow-up questions that Prof. Gluck had in mind that we didn't get to. I did bumble into the answers for HG2d, HG2e, and HG2h while attempting HG2a.

2b) What is the fundamental group of the space of isometries of a round $\mathbb{RP}^n$?

2c) Think of $SO(n)$ as a manifold embedded in $\mathbb{R}^{n^2}$ and give it the induced metric. Is this metric bi-invariant? Homogeneous? Isotropic?

2d) Check that $SO(2)$ is a circle. How big?

2e) Check that $SO(3)$ is a round $\mathbb{RP}^3$. How big?

2f) Use the formula for sectional curvatures of a bi-invariant metric on a Lie group to compute the sectional curvatures of $SO(3)$ and check against the answer to (e).

2g) Between what two limits do the sectional curvatures of $SO(n)$ vary?

2h) Describe the geodesics in $SO(n)$. How do you know this?

Differential Geometry (CC)

1a) Let M be an orientable complete closed 2-manifold with positive curvature. Must any two closed geodesics intersect?

1b) Now let M be an orientable complete closed 3-manifold with positive sectional curvature. Must any two closed geodesics intersect?

2a) Let M be a 2-torus with one point removed. Is there a metric g on M such that (M, g) is complete?

2b) Is there a metric g on M such that (M, g) is complete and has positive curvature?

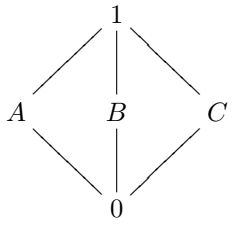
* Problem 2 was previously posed during Mohammed Hindawi's oral exam.

Combinatorics (RP)

1a) Write a generating function for the number of involutions on n letters.

1b) What are the asymptotics for the number of involutions on n letters?

2a) Is the following poset distributive?



2b) Draw a non-trivial distributive lattice.

2c) What is a Möbius function and how would you compute it on the lattice you drew?

2d) What is a derangement? Count the number of derangements on n letters.

3) Consider the exponential generating function

$$f(z) = \frac{\exp(-z/2 - z^2/4)}{\sqrt{1-z}}.$$

What are the asymptotics of its coefficients (which count 2-regular graphs on n vertices)?